THE CONSISTENCY STRENGTH OF THE GENERALIZED CONTINUUM HYPOTHESIS FAILING AT A MEASURABLE CARDINAL

by

Connor Watson

A Thesis Submitted to the Faculty of

The Charles E. Schmidt College of Science

in Partial Fulfillment of the Requirements for the Degree of

Master of Science

Florida Atlantic University

Boca Raton, FL

May 2025

THE CONSISTENCY STRENGTH OF THE GENERALIZED CONTINUUM HYPOTHESIS FAILING AT A MEASURABLE CARDINAL

by

Connor Watson

This thesis was prepared under the direction of the candidate's thesis advisor, Dr. Robert Lubarsky, Department of Mathematics and Statistics, and has been approved by the members of his supervisory committee. It was submitted to the faculty of the Charles E. Schmidt College of Science and was accepted in partial fulfillment of the requirements for the degree of Master of Science.

SUPERVISORY COMMITTEE:

_____
Robert Lubarsky, Ph.D.
Thesis Advisor

_____
James Cummings, Ph.D.

_____
James Hanson, Ph.D.

_____
Zvi Rosen, Ph.D.

_____
Yuan Wang, Ph.D.
Chair, Department of Mathematics and
Statistics

_____
Valery Forbes, Ph.D.
Dean, The Charles E. Schmidt College of
Science

_____
Robert Stackman, Jr., Ph.D.
Dean, Graduate College

_____
Date

iii

ACKNOWLEDGEMENTS

There is insufficient space here to enumerate all of the ways in which the following individuals and groups have been instrumental in the creation of this work.

My advisor Robert Lubarsky, for fostering my love for set theory and for giving me the first space I knew where *all* ideas are welcome. Thank you for broadening my horizons and allowing me to share mine. You have made me the student I am today and I am eternally grateful for that. I hope I have made you proud.

James Cummings, for taking a nobody under your wing, for your countless insightful discussions, and for giving me a taste of student life at Carnegie Mellon University. A set theory student navigating a university with only one logician was an impossible task, and your invaluable expertise made this thesis possible.

James Hanson, for everything from the time you spent to be on my thesis committee to your 3am ramblings about constructive mathematics.

All of my instructors who have, in their own unique ways, shaped me into who I am today. Special thanks to Timothy Ford, Zvi Rosen, and Tomas Schonbek for the time you spent going above and beyond to show me the world of mathematics. Additional thanks to Stephen Anand, Lisa Holland, and Evan Rogovin: I hope we can continue to stay in touch so you can see how much I have grown since we first met.

The FAU Math Department, for bending over backwards time and time again to make a program that works for me. I have had one of the most nontraditional

ABSTRACT

Author: Connor Watson

Title: The Consistency Strength of the Generalized Continuum Hypothesis Failing at a Measurable Cardinal

Institution: Florida Atlantic University

Thesis Advisor: Dr. Robert Lubarsky

Degree: Master of Science

Year: 2025

This expository paper investigates the equiconsistency of the GCH failing at a measurable cardinal with the existence of a cardinal κ of Mitchell order κ^{++}.

The upper bound of this equiconsistency follows in two parts: Assuming the existence of a model of $o(\kappa) = \kappa^{++}$, one can first use an argument of Gitik to force the existence of an elementary embedding satisfying certain closure conditions, then use a forcing due to Woodin to force the failure of GCH at κ while preserving the measurability of κ. It is this Woodin result which this thesis focuses on in the upper bound.

The lower bound of this equiconsistency is an inner-model-theoretic argument due to Mitchell, where one can show that assuming the GCH fails at a measurable cardinal, then K, the so-called 'core model below $o(\kappa) = \kappa^{++}$' exists. This thesis aims to bridge a gap in the literature by providing a much-needed approachable introduction to inner model theory at the level of $o(\kappa) = \kappa^{++}$ for the non-specialist.

Mitchell's argument that the GCH failing at a measurable cardinal implying the existence of a model of $o(\kappa) = \kappa^{++}$ is then given.

To Ozzy

THE CONSISTENCY STRENGTH OF THE GENERALIZED CONTINUUM HYPOTHESIS FAILING AT A MEASURABLE CARDINAL

INTRODUCTION

Set theory arguably began with Cantor's theorem: $|P(X)| > |X|$ for any set X, which started the investigation of the continuum function $\kappa \mapsto 2^\kappa$. Later, the Generalized Continuum Hypothesis (GCH), the assertion that $2^\kappa = \kappa^+$ for every cardinal κ, was proven independent of the ZFC axioms by Gödel and Cohen, with Godel showing that the GCH holds in the constructible universe, while Cohen developed the method of forcing to build a model of ZFC in which $2^{\aleph_0} = \aleph_2$.

A flurry of activity ensued to determine the possible behavior of the continuum function. Easton showed that on the class of regular cardinals, the continuum function $\kappa \mapsto 2^\kappa$ can consistently behave in any way barring restrictions with respect to monotinicity or König's theorem that $\mathrm{cf}(2^\kappa) > \kappa$. Thus, the focus turned to the case of singular cardinals.

One version of the Singular Cardinals Hypothesis (SCH) states that the GCH holds for singular strong limit cardinals. At the time of Easton's discovery, it was reasonable to think that a similar theorem could be proven about the continuum function on singular cardinals, despite Easton's method not being able to be used. This would prove to be misguided, as Silver showed that the GCH cannot first fail at a singular cardinal of uncountable cofinality. This led to more investigation of the problem; three main approaches were taken in investigating SCH: pcf theory, inner model theory, and the combination of forcing and large cardinals.

Shelah developed his purely combinatorial pcf (*possibile cofinalities*) theory in a series of papers, culminating in his celebrated theorem that if α is a limit ordinal with $|\alpha|^{\mathrm{cf}(\alpha)} < \aleph_\alpha$, then $(\aleph_\alpha)^{\mathrm{cf}(\alpha)} < \aleph_{|\alpha|^+ 4}$, so in particular $2^{\aleph_\omega} < \aleph_{\omega_4}$ assuming $\aleph_\omega$ is a strong limit cardinal.

The methods of inner model theory also proved useful here. Failures of SCH of all different kinds have been shown to imply the existence of inner models containing large cardinals with high consistency strength, which give lower bounds for these consistency strength of these failures. The theories of Prikry, Radin, and Magidor forcing also play a role here.

Finally, combining Cohen's method of forcing with large cardinal axioms, we achieve upper bounds on consistency strength bounds on the failure of the SCH. Silver's original theorem showed that GCH cannot first fail at a singular cardinal of uncountable cofinality: in particular, it cannot first fail at a measurable cardinal. A natural question arises: can it fail at a measurable cardinal, and if so, what is the large cardinal strength of this assertion? The answer turned out to be yes, it is possible, and work of Gitik, Magidor, Mitchell, Silver, and Woodin established the equiconsistency of GCH failing at a measurable cardinal with the existence of a cardinal κ of Mitchell order κ^{++}. It is this result which this thesis focuses on.

This result illustrates a common theme when proving equiconsistency results: an upper bound for the consistency strength is often given using the method of forcing, where a generic extension in which the desired property holds is constructed assuming the existence of the large cardinal, and the consistency bound is proven optimal using inner model theoretic approaches. The structure of this thesis aims to highlight this interplay.

We outline the approach taken in this paper below. We start with a preliminary section outlining the necessary theory of forcing, elementary embeddings, and large cardinals. In Chapter 2, we present Silver's proof to give a weak upper bound on the consistency strength of the GCH failing at a measurable cardinal (κ^{++}-supercompactness), followed by Woodin's proof assuming the existence of an elementary embedding $j : V \to M$ with $^{\kappa}M \subseteq M$ and $(\kappa^{++})^M = \kappa^{++}$. That the existence of an elementary embedding of this form can be forced over a certain model with a car-

dinal κ of Mitchell order κ^{++} is due to Gitik, which would complete the upper bound of our desired equiconsistency, but we omit this result. Finally, Chapter 3 gives an introduction to Mitchell's core model K below $o(\kappa) = \kappa^{++}$, and ends with his theorem that the GCH failing at a measurable cardinal implies the existence of an inner model with a cardinal of Mitchell order κ^{++}, which establishes the equiconsistency.

As far as prerequisite knowledge is concerned, some prior acquaintance with iterated forcing, measurable cardinals and elementary embeddings, and sharps would be highly desirable, if not actually necessary. Nevertheless, these topics appear in the preliminary chapter in order to keep this thesis as self-contained as possible. A knowledge of forcing is absolutely necessary.

CHAPTER 1

PRELIMINARIES

1.1 FORCING

In this section, we review some basic definitions and prove some basic facts about forcing. Our treatment mostly follows [8].

A *notion of forcing* is officially a triple $(\mathbb{P}, \leq, \mathbf{1}_{\mathbb{P}})$ where $\mathbb{P}$ is a partial order with top element $\mathbf{1}_{\mathbb{P}}$ and $\leq$ is the ordering on $\mathbb{P}$. We follow the convention that $q \leq p$ means that q is *stronger* than p.

The class of $\mathbb{P}$-names, denoted $V^{\mathbb{P}}$, is defined by transfinite recursion, by letting

$$
\begin{aligned}
V_0^{\mathbb{P}} &= \emptyset, \\
V_{\alpha+1}^{\mathbb{P}} &= \mathcal{P}(V_\alpha^{\mathbb{P}} \times \mathbb{P}), \\
V_\lambda &= \bigcup_{\beta < \lambda} V_\beta^{\mathbb{P}} \quad \text{for } \beta \text{ a limit ordinal.}
\end{aligned}
\tag{1.1}
$$

Finally, we let

$$
V^{\mathbb{P}} = \bigcup_{\alpha \in \mathrm{ON}} V_\alpha.
$$

If M is a transitive model of ZFC and $\mathbb{P} \in M$, then $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$.

Next, we say a subset D of $\mathbb{P}$ is *dense* if for every p in $\mathbb{P}$ there is a $q \leq p$ and *open dense* if it is dense and in addition, $p \in D$ and $q \leq p$ imply $q \in D$. If M is a transitive model of set theory with $\mathbb{P} \in M$, then we say a filter G is $\mathbb{P}$-*generic over* M (or, if the context is clear, $\mathbb{P}$-*generic* or just *generic*) if $G \cap D \neq \emptyset$ for all $D \in M$ dense in $\mathbb{P}$. It is easy to show that if M is countable and $\mathbb{P} \in M$, then there is a G which is $\mathbb{P}$-generic over M, thus for the remainder of this section, M will denote a countable

transitive model of set theory if unspecified. While the consistency of ZFC does not prove the existence of a transitive model of ZFC, it is a standard fact in the theory of forcing that the transitivity assumption is merely a convenience and not actually necessary. For a discussion of the metamathematics behind this, see [8].

Given a filter G which is $\mathbb{P}$-generic over M and $\dot{x} \in M^{\mathbb{P}}$, we define the *interpretation of the name $\dot{x}$ with respect to G* recursively on $M^{\mathbb{P}}$, denoted $i_G(\dot{x})$, by letting

$$i_G(\dot{x}) = \{i_G(\dot{y}) : \exists p \in G((\dot{y}, p) \in \dot{x})\}.$$

Now we can define the forcing extension $M[G] = \{i_G(\dot{x}) : \dot{x} \in M^{\mathbb{P}}\}$.

We define the canonical name for a set $x \in M$, denoted $\check{x}$, recursively, by setting $\check{x} = \{(\check{y}, \mathbf{1}_{\mathbb{P}}) : y \in x\}$.

Recall that the canonical name for the generic G is $\Gamma = \{(\check{p}, p) : p \in \mathbb{P}\}$.

If φ is a formula with n free variables in the language of set theory, $\mathbb{P}$ is a partial order in a countable transitive model M, $p \in \mathbb{P}$, and $\dot{x}_1, ..., \dot{x}_n \in M^{\mathbb{P}}$, then we say that *$p$ forces φ*, written $p \Vdash \varphi(i_G(\dot{x}_1), ..., i_G(\dot{x}_n))$, if $M[G] \vDash \varphi(i_G(\dot{x}_1), ..., i_G(\dot{x}_n))$ for every generic G such that $p \in G$.

We are now ready to state the main theorems of forcing.

Theorem 1.1.1 (Forcing Theorem)**.** *Let M be a countable transitive model of ZFC and $\mathbb{P} \in M$ be a notion of forcing. Let φ be a formula in the language of set theory with n free variables, and let $\dot{x}_1, ..., \dot{x}_n \in M^{\mathbb{P}}$. Then there is a relation $\Vdash'$ uniformly definable in M such that for all $p \in \mathbb{P}$, $p \Vdash \varphi(i_G(\dot{x}_1), ..., i_G(\dot{x}_n)) \iff (p \Vdash' \varphi(i_G(\dot{x}_1), ..., i_G(\dot{x}_n)))^M$.*

Theorem 1.1.2 (Generic Model Theorem)**.** *With the same hypotheses as Theorem 1,*

$$M[G] \vDash \varphi(i_G(\dot{x}_1), ..., i_G(\dot{x}_n)) \iff \exists p \in G(p \Vdash \varphi(i_G(\dot{x}_1), ..., i_G(\dot{x}_n)),$$

for all G which are $\mathbb{P}$-generic over M.

Proof. By induction on φ. $\hfill\square$

In summary, Theorem 1.1.1 and Theorem 1.1.2 say that the forcing relation $\Vdash$ is uniformly definable in M, and that every statement which is true in a generic extension is forced by a condition.

Next, we review several conditions which ensure the preservation of cardinals and cofinalities in generic extensions.

Definition 1.1.3. Let $\mathbb{P}$ be a partial order.

1. If $p, q \in \mathbb{P}$, then p and q are *compatible*, written $p \parallel q$, if they have a common extension, i.e. if there is an $r \in \mathbb{P}$ such that $r \le p \wedge r \le q$.

2. If $p, q \in \mathbb{P}$, then p and q are *incompatible*, written $p \perp q$, if they are not compatible.

3. $\mathbb{P}$ is *separative* if for all $p, q \in \mathbb{P}$ with $p \not\le q$, there is an $r \le p$ with $r \perp q$.

4. An *antichain* in $\mathbb{P}$ is a subset X of $\mathbb{P}$ such that for every $p, q \in X$, $p \ne q \implies p \perp q$.

5. If κ is an uncountable cardinal, then $\mathbb{P}$ has the *κ-chain condition*, abbreviated κ-c.c., if every antichain in $\mathbb{P}$ has size less than κ.

6. $\mathbb{P}$ has the *countable chain condition*, abbreviated c.c.c., if it has the $\aleph_1$-chain condition.

7. $\mathbb{P}$ is *κ-closed* if every sequence in $\mathbb{P}$ of length less than κ has a lower bound.

8. A subset $D \subseteq \mathbb{P}$ is called *directed* if for every $p, q \in D$, there is an $r \in D$ such that $p \le r$ and $q \le r$. $\mathbb{P}$ is *κ-directed closed* if every directed set of size less than κ has a lower bound.

9. $\mathbb{P}$ is *κ-distributive* if the intersection of less than κ open dense sets in $\mathbb{P}$ is dense.

10. $\mathbb{P}$ has the κ-*Knaster* condition if and only if for every sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of conditions in $\mathbb{P}$, there is an unbounded set $X \subset \kappa$ so that $p_\alpha \parallel p_\beta$ whenever $\alpha, \beta \in X$.

Remark 1.1.4. In the literature, there is no consensus on whether (κ-(directed) closure, or κ-distributivity) refer to subsets of $\mathbb{P}$ of size less than κ or to subsets of size less than or equal to κ. The reader should notice that our conventions refer to subsets of $\mathbb{P}$ of size *less than* κ.

Lemma 1.1.5. If $\mathbb{P}$ is κ-closed, then $\mathbb{P}$ is κ-distributive.

Proof. Suppose $\lambda < \kappa$ and $\langle D_\alpha : \alpha < \lambda \rangle$ is a collection of open dense sets in $\mathbb{P}$. We wish to show that $D = \bigcap_{\alpha < \lambda} D_\alpha$ is dense in $\mathbb{P}$. Let $p \in \mathbb{P}$, and using the κ-closure of $\mathbb{P}$, construct a sequence $\langle p_\alpha : \alpha < \lambda \rangle$ of conditions such that for every $\alpha < \lambda$,

1. $p \geq p_\alpha$,

2. $p_\alpha \geq p_\beta$ for every $\beta < \alpha$, and

3. $p_\alpha \in D_\alpha$.

By the κ-closure of $\mathbb{P}$ once again, let q be a lower bound for the sequence $\langle p_\alpha : \alpha < \lambda \rangle$. This is a condition $q \in D$ with $q \leq p$, and so D is dense. Hence, $\mathbb{P}$ is κ-distributive. $\qquad\square$

Now we know that κ-closure implies κ-distributivity, but it will also be useful for later to define a property strictly in-between the two.

Definition 1.1.6. Let $\mathbb{P}$ be a notion of forcing and κ a regular cardinal. Then $\mathbb{P}$ is κ-*strategically closed* if and only if player I has a winning strategy in the following "κ-strategic closure game on $\mathbb{P}$":

The κ-strategic closure game on $\mathbb{P}$ is a two-player game of perfect information, where players I and II alternate playing conditions in $\mathbb{P}$ to build a κ-sequence. Player

I plays the even stages and limit stages, and player II plays the odd stages. Player I must play $1_\mathbb{P}$ at move zero or they lose. Let p_α be the condition played at move α. The player who played p_α loses immediately unless $p_\alpha \leq p_\beta$ for all $\beta < \alpha$. If neither player loses at any stage $\alpha < \kappa$, then player I wins.

We have seen that κ-closure implies κ-distributivity, and the following implications are similarly easy:

$$\kappa\text{-directed closure} \implies \kappa\text{-closure} \implies \kappa\text{-strategic closure} \implies \kappa\text{-distributivity}.$$

Remark 1.1.7. One can find forcing posets which show that each of these closure conditions are actually distinct.

1. The forcing which adds a Kurepa tree at an inaccessible cardinal κ is κ-closed but not κ-directed closed.

2. The forcing which adds a non-reflecting stationary set to a regular cardinal $\kappa \geq \aleph_2$ is κ-strategically closed but not κ-closed.

3. The forcing which shoots a club through a stationary costationary subset of $\aleph_1$ is $\aleph_1$-distributive but not $\aleph_1$-strategically closed.

Theorem 1.1.8. *If M is a countable transitive model of ZFC, $\kappa \in M$, ($\mathbb{P}$ is κ-distributive)^M, and G is $\mathbb{P}$-generic over M, then $^{<\kappa}M \cap M[G] \subseteq M$.*

Proof. Note that without loss of generality, we may replace density in the definition of distributivity with density beneath a given $p \in \mathbb{P}$.

Let $f : \gamma \to M$ with $\gamma < \kappa$ and $f \in M[G]$. Let $\dot{f} \in M$ with $f = i_G(\dot{f})$, and let $p \in G$ be a condition so that $p \Vdash$ "$\dot{f}$ is a function on $\check\gamma$.

For $\alpha < \gamma$, let

$$U_\alpha = \{q \leq p : \exists x \; q \Vdash \dot{f}(\check\alpha) = \check x\}.$$

Then each U_α is open and dense below p.

By Theorem 1.1.1, $\langle U_\alpha : \alpha < \gamma \rangle \in M$.

Now let $D = \bigcap_{\alpha < \gamma} U_\alpha$. Then $D \in M$ and D is dense in the set $\{q \in \mathbb{P} : q \leq p$ by κ-distributivity. Let $q \in G \cap D$, which exists by genericity. Then, for each x and each $\alpha < \gamma$, we have that $f(\alpha) = x \iff q \Vdash \dot{f}(\check{\alpha}) = \check{x}$. By Theorem 1.1.1 once again, we see that $f \in M$. $\qquad\square$

Theorem 1.1.8 tells us that forcing with κ-distributive forcing preserves κ. The following theorem will also prove to be useful in computing cardinals in generic extensions.

Theorem 1.1.9. *If (κ is a regular cardinal of uncountable cofinality)^M and $\mathbb{P} \in M$ is a κ-c.c. poset, then for every ordinal $\mu \geq \kappa$,*

1. $|\mu|^M = |\mu|^{M[G]}$

2. *If* $\mathrm{cf}(\mu) \geq \kappa$, *then* $(\mathrm{cf}(\mu))^M = (\mathrm{cf}(\mu))^{M[G]}$.

In particular, if ($\mathbb{P}$ has the countable chain condition)^M, then forcing with $\mathbb{P}$ preserves all cardinals and cofinalities.

Proof. The proof of this fact uses the standard 'nice names' argument, the details of which can be found in [5] or [8] $\qquad\square$

We now recall the definitions of Cohen forcing, the collapse forcing, and the Lévy collapse forcing.

Definition 1.1.10. Let κ be a regular cardinal and let λ be any cardinal. Define the poset

$$\mathrm{Add}(\kappa, \lambda) = \{p : p \text{ is a partial function } \kappa \times \lambda \to 2 \text{ with } |p| < \kappa\}.$$

Conditions are ordered by reverse inclusion, that is, $p \leq q$ iff $p \supseteq q$.

Definition 1.1.11. Let κ be a regular cardinal and let λ be a cardinal with $\lambda \geq \kappa$. The *collapse forcing* to collapse λ to κ is the partial order

$$\mathrm{Col}(\kappa, \lambda) = \{p : p \text{ is a partial function } \kappa \to \lambda \text{ and } |p| < \kappa\}.$$

Conditions are ordered by reverse inclusion, that is, $p \leq q$ iff $p \supseteq q$.

Definition 1.1.12. Let κ be a regular cardinal and let λ be a cardinal with $\lambda > \kappa$. Then the *Lévy collapse* is the partial order

$$\mathrm{Col}(\kappa, < \lambda) = \{p : \mathrm{dom}(p) \subseteq \lambda \times \kappa, \ |\mathrm{dom}(p)| < \kappa, \text{ and } p(\alpha, \xi) < \xi \text{ for each } (\alpha, \xi) \in \mathrm{dom}(p)\}.$$

Theorem 1.1.13 (Cohen Forcing). *Let κ be a regular cardinal, let λ be any cardinal, and let $\mathbb{P} = \mathrm{Add}(\kappa, \lambda)$. Then*

1. *$\mathbb{P}$ is κ-closed.*

2. *$\mathbb{P}$ has the $(2^{<\kappa})^+$-c.c.*

3. *If M is a countable transitive model of ZFC, G is $\mathbb{P}$-generic over M, and*

$$M \vDash \text{``}\kappa^{<\kappa} = \lambda^\kappa = \kappa,\text{''}$$

   then all cardinals are preserved in $M[G]$, and $M[G] \vDash \text{``}2^\kappa = \lambda.\text{''}$

Theorem 1.1.14 (Collapse Forcing). *Let κ be a regular cardinal, let $\lambda \geq \kappa$ be any cardinal, and let $\mathbb{P} = \mathrm{Col}(\kappa, \lambda)$. Then*

1. *$\mathbb{P}$ is κ-closed,*

2. *$\mathbb{P}$ has the $(\lambda^{<\kappa})^+$-c.c, and*

3. *if M is a countable transitive model of ZFC and G is $\mathbb{P}$-generic over M, then*

$$M[G] \vDash |\lambda| = \kappa.$$

   Moreover, $M[G]$ preserves cardinals $\leq \kappa$, and if $\lambda^{<\kappa} = \lambda$, then $M[G]$ preserves cardinals $> \lambda$.

Theorem 1.1.15 (Lévy Collapse). *Let κ be a regular cardinal, let $\lambda > \kappa$ be an inaccessible cardinal, and let $\mathbb{P} = \mathrm{Col}(\kappa, < \lambda)$. Then*

1. *$\mathbb{P}$ is κ-closed.*

2. *$\mathbb{P}$ has the λ-c.c.*

3. *If M is a countable transitive model of ZFC and G is $\mathbb{P}$-generic over M, then*

$$M[G] \models \lambda = \kappa^+.$$

Easton's Lemma 1.1.16 will also be of significant use later and we record its proof here.

Lemma 1.1.16. Let κ be a regular cardinal, and let $\mathbb{P}$ and $\mathbb{Q}$ be notions of forcing. Suppose $\mathbb{P}$ is κ^+-c.c. and $\mathbb{Q}$ is κ^+-closed. Then

$$\Vdash_{\mathbb{P}} \text{``}\mathbb{Q} \text{ is } \kappa^+\text{-distributive."}$$

Proof. Suppose G is a V-generic filter on $\mathbb{P}$ and $\langle D_\alpha : \alpha < \kappa \rangle$ is a sequence of dense open subsets of $\mathbb{Q}$ in $V[G]$. It suffices to show that for any $q \in \mathbb{Q}$, there is $q' \leq q$ so that

$$q' \in \bigcap_{\alpha < \kappa} D_\alpha.$$

Let $\tau \in V$ be a name for $\langle D_\alpha : \alpha < \kappa \rangle$, and let $p \in G$ be a condition so that

$$p \Vdash_{\mathbb{P}} \text{``}\tau \text{ is a sequence of } \check{\kappa} \text{ many open dense sets in } \check{\mathbb{Q}}\text{."}$$

We will find a $q' \leq q$ so that

$$p \Vdash_{\mathbb{P}} \text{``}\check{q}' \in \bigcap_{\alpha < \kappa} \tau(\alpha)\text{."}$$

Work in V. We will construct by recursion on α a decreasing sequence $\langle q_\alpha : \alpha < \kappa \rangle$ in $\mathbb{Q}$ so that $q_0 = q$, and for every $\alpha < \kappa$, $p \Vdash \check{q}_{\alpha+1} \in \tau(\check{\alpha})$. This is sufficient, because

11

by the κ-closure of $\mathbb{Q}$, we may take q' to be a lower bound for $\langle q_\alpha : \alpha < \kappa \rangle$, which would finish the proof.

To this end, let $q_0 = q$. First, we consider the successor case of the recursion. Assume q_α has been defined. By recursion on $\gamma < \kappa$, for as long as possible, we define conditions $p_\gamma \in \mathbb{P}$ and $r_\gamma \in \mathbb{Q}$ so that

1. $p_\gamma \leq p$ and $r_\gamma \leq q_\alpha$,

2. if $\eta < \gamma$, then $p_\eta \perp p_\gamma$ and $r_\eta \geq r_\gamma$, and

3. $p_\gamma \Vdash$ "$\check{r}_\eta \in \tau(\alpha)$."

Suppose $\langle p_\eta : \eta < \gamma \rangle$ and $\langle r_\eta : \eta < \gamma \rangle$ have been defined.

If $\langle p_\eta : \eta < \gamma \rangle$ is not a maximal antichain below p, let p' be a condition beneath p so that $p' \perp p_\eta$ for every $\eta < \gamma$. By the κ-closure of $\mathbb{Q}$, pick an r' so that $r' \leq r_\eta$ for every $\eta < \gamma$. Then let p_γ and r_γ be conditions beneath p' and r' respectively so that

$$p_\gamma \Vdash \text{``}\check{r}_\gamma \in \tau(\check{\alpha})\text{.''}$$

The conditions p_γ and r_γ can be chosen in this way since $p \Vdash \tau(\check{\alpha})$ is dense.

There exists some ξ so that $\langle p_\alpha : \alpha < \xi \rangle$ is a maximal antichain beneath p, and it is at this point the process ends. We then let $q_{\alpha+1}$ be a lower bound of $\langle r_\alpha < \xi \rangle$. This condition satisfies the requirement that

$$p \Vdash \text{``}\check{q}_{\alpha+1} \in \tau(\check{\alpha})\text{.''}$$

If α is a limit ordinal, we simply use the closure of $\mathbb{Q}$ and let q_α be a lower bound of $\langle q_\beta : \beta < \alpha \rangle$.

As discussed before, we let q' be a lower bound of $\langle q_\alpha : \alpha < \kappa \rangle$, and this satisfies that

$$p \Vdash_{\mathbb{P}} \text{``}\check{q}' \in \bigcap_{\alpha < \kappa} \tau(\alpha)\text{.''}$$

Thus, $\Vdash_{\mathbb{P}}$ "$\mathbb{Q}$ is κ-distributive", which completes the proof of Lemma 1.1.16. $\square$

1.2 ITERATED FORCING

In this section we review the basics of iterated forcing. Our treatment mostly follows [1] and [5], where we recommend the reader go for further reading.

Definition 1.2.1. Suppose $\mathbb{P}$ is a notion of forcing and $\dot{\mathbb{Q}}$ is a name so that $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is a notion of forcing. Then the two step iteration $\mathbb{P} * \dot{\mathbb{Q}} = \{(p, \dot{q}) : p \in \mathbb{P} \wedge \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}\}$, where $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if and only if $p_1 \leq p_2$ and $p_1 \Vdash \dot{q}_1 \leq \dot{q}_2$.

It is not immediately clear what $\mathbf{1}_{\mathbb{Q}}$ should be. We may implicitly arrange so that $\mathbf{1}_{\mathbb{Q}} \in V$, so we will assume that $\mathbf{1}_{\mathbb{Q}}$ is an element of the ground model and that $\dot{\mathbf{1}}_{\mathbb{Q}}$ is determined by $\mathbf{1}_{\mathbb{P}}$.

Forcing with the two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is the same as first forcing with $\mathbb{P}$ and then forcing with $\mathbb{Q}$. This is made precise in the following theorem:

Theorem 1.2.2.   *1. Suppose G is $\mathbb{P}$-generic over V and H is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$. Then $I := \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \in \mathbb{P} \wedge i_G(q) \in H\}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V.*

2. Suppose I is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V. Then $G := \{p \in \mathbb{P} : \exists q (p, \dot{q}) \in I\}$ is $\mathbb{P}$-generic over V and $H := \{i_G(\dot{q}) : \exists p (p, \dot{q}) \in I\}$ is $i_G(\dot{\mathbb{Q}})$-generic over $V[G]$.

In each case, we have that $V[I] = V[G][H]$.

Proof. For each case, we only check genericity.

1. Suppose D is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. Define A in $V[G]$ so that $q^{V[G]} \in A$ if and only if $\exists p \in G (p, \dot{q}) \in D$, and let $\check{A}$ be the canonical name for A. We claim $\Vdash_{\mathbb{P}} \check{A}$ is dense in $\mathbb{Q}$. Suppose $p \in \mathbb{P}$ and $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$. By density, we can find some $(p_1, q_1) \in D$ such that $(p_1, q_1) \leq (p, q)$. But notice $p_1 \Vdash$ "$\dot{q}_1 \in \check{A}$ and $\dot{q}_1 \leq \dot{q}$". Thus $\{p \in \mathbb{P} : p \Vdash \exists q \leq \dot{q}\, q \in \check{A}\}$ is dense in $\mathbb{P}$. Now choose $\dot{q}^{V[G]} \in A \cap H$ and let $p \in G$ force that $\dot{q} \in \check{A}$. Then $(p, \dot{q}) \in I \cap D$.

2. Suppose $D = i_G(\dot{D}) \subseteq \mathbb{Q}$ is a dense subset of $\mathbb{Q}$ in $V[G]$, and let $p_0 \in G$ be a condition which forces this. Let $E = \{(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}} : p \leq p_0 \text{ and } p \Vdash \dot{q} \in \dot{D}\}$. It is easy to see that D' is dense below $(p_0, \mathbf{1}_{\mathbb{Q}})$, so there exists $(p, \dot{q}) \in I \cap D'$. But then $q \in I \cap D$, so $I \cap D \neq \emptyset$. $\qquad\square$

It is not hard to see how an iteration of length 3 would be defined: simply force with $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$. This motivates us to define a more general iteration, which works into the transfinite.

Definition 1.2.3. Suppose $\alpha \geq 2$. Then $\mathbb{P}_\alpha$ is an α-*iteration* if $\mathbb{P}_\alpha$ is a set of α-sequences satisfying:

1. If $\alpha = \beta + 1, \beta \geq 1$, then $\mathbb{P}_\beta = \{p \restriction \beta : p \in \mathbb{P}_\alpha\}$ is a β-iteration and there is $\dot{\mathbb{Q}}_\beta$ so that $\mathbf{1}_{\mathbb{P}_\beta} \Vdash \dot{\mathbb{Q}}_\beta$ is a partial ordering, and $p \in \mathbb{P}_\alpha$ if and only if $p \restriction \beta \in \mathbb{P}_\beta$ and $\mathbf{1}_{\mathbb{P}_\beta} \Vdash p(\beta) \in \dot{\mathbb{Q}}_\beta$. Moreover, $p \leq q$ if and only if $p \restriction \beta \leq q \restriction \beta$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \leq q(\beta)$, so that $\mathbb{P}_\alpha \cong \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$.

2. If α is a limit ordinal then for every $\beta < \alpha, \mathbb{P}_\beta = \{p \restriction \beta : p \in \mathbb{P}_\alpha\}$ is a β-iteration, and

   (a) If $\beta < \alpha, p \in \mathbb{P}_\alpha, q \in \mathbb{P}_\beta$, and $q \leq p \restriction \beta$, then $r \in \mathbb{P}_\alpha$, where $r \restriction \beta = q$ and $r(\gamma) = p(\gamma)$ for every $\beta \leq \gamma < \alpha$.

   (b) If $p, q \in \mathbb{P}_\alpha$, $p \leq q$ if and only $\forall \beta < \alpha, p \restriction \beta \leq q \restriction \beta$.

   (c) $\mathbf{1}_{\mathbb{P}_\alpha}$ is the α-sequence with $\mathbf{1}_{\mathbb{P}_\alpha}(\gamma) = \mathbf{1}_{\dot{\mathbb{Q}}_\gamma}$.

This definition coincides with Definition 1.2.1 in the case of $\alpha = 2$, and also uniquely determines $\mathbb{P}_\beta$ at successor stages. However, $\mathbb{P}_\gamma$ is not determined by $\langle \mathbb{P}_\beta : \beta < \gamma \rangle$ when γ is a limit ordinal. There are several options, two of which will be of particular importance in the forcing constructions to follow.

Definition 1.2.4. The *direct limit* of $\langle \mathbb{P}_\beta : \beta < \gamma \rangle$ is the set of all p such that for some $\beta < \gamma$,

14

1. $p \restriction \beta \in \mathbb{P}_\beta$ and

2. For every ordinal ξ, $\beta \le \xi < \gamma \implies p(\xi) = 1_{\dot{\mathbb{Q}}_\xi}$.

Definition 1.2.5. If γ is a limit ordinal, then the *inverse limit* of $\langle \mathbb{P}_\beta : \beta < \gamma \rangle$ is the set of all p such that $p \restriction \beta \in \mathbb{P}_\beta$ for all $\beta < \alpha$.

Next, recall the definition of the support of a condition:

Definition 1.2.6. If $\mathbb{P}_\alpha$ is an α-iteration and $p \in \mathbb{P}_\alpha$, then the *support* of p, denoted supp(p), is the set $\{\beta < \alpha : p(\beta) \neq 1_{\dot{\mathbb{Q}}_\beta}\}$.

One particular kind of iteration which will be used in all forthcoming forcing arguments is the Easton iteration, which is useful for guaranteeing a balance between preserving chain conditions and closures.

Definition 1.2.7. $\mathbb{P}_\alpha$ is an iteration with *Easton support*, or an Easton iteration, if for every limit ordinal $\gamma \le \alpha$, $\mathbb{P}_\gamma$ is a direct limit if γ is regular, and an inverse limit otherwise.

Theorem 1.2.8. *If $\beta < \alpha$, G_α is $\mathbb{P}_\alpha$-generic over V, and $G_\beta := \{p \restriction \beta : p \in G_\alpha\}$, then G_β is $\mathbb{P}_\beta$-generic over V.*

Proof. Once again we only check the density condition. Suppose D is dense in $\mathbb{P}_\beta$. For each $p \in \mathbb{P}_\beta$, let $p' \in \mathbb{P}_\alpha$ be such that $p' \restriction \beta = p$ and $p'(\gamma) = 1_{\dot{\mathbb{Q}}_\gamma}$ if $\beta \le \gamma < \alpha$. Let $D' = \{q \in \mathbb{P}_\alpha : (\exists p \in D) q \le p'\}$. We claim D' is dense in $\mathbb{P}_\alpha$. If $r \in \mathbb{P}_\alpha$ then $r \restriction \beta \in \mathbb{P}_\beta$, so there is $p \in D$ such that $p \le r \restriction \beta$. If $q \in \mathbb{P}_\alpha$ is defined so that $q \restriction \beta = p$ and $q(\gamma) = r(\gamma)$ for $\beta \le \gamma < \alpha$, then $q \le p'$ and so $q \in D'$. Hence, D' is dense in $\mathbb{P}_\alpha$. Then, $G_\alpha \cap D' \neq \emptyset$, so there's a $p \in D$ with $p' \in G_\alpha$. But then $p \in G_\beta \cap D$, so $G_\beta \cap D \neq \emptyset$. $\square$

Theorems 1.2.9 and 1.2.10 give sufficient criteria for chain condition and closure properties to be preserved in forcing iterations. We refer the reader to [1] or [5] for the details.

Theorem 1.2.9.

1. *If $\mathbb{P}$ has the κ-c.c. and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ has the κ-c.c.", then $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-c.c.*

2. *If $\mathbb{P}_\alpha$ is an α-iteration, and direct limits are taken at stage α and at stationarily many $\beta < \alpha$, and $\mathbb{P}_\beta$ has the κ-c.c. for every $\beta < \alpha$, then $\mathbb{P}_\alpha$ has the κ-c.c.*

Theorem 1.2.10. *Let κ be an infinite cardinal.*

1. *If $\mathbb{P}$ is κ-directed closed and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed", then $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-directed closed.*

2. *If $\mathbb{P}_\alpha$ is an α-iteration for some α with $\mathrm{cf}(\alpha) > \kappa$, $\mathbb{P}_\beta$ is κ-directed closed for every $\beta < \alpha$, and a direct limit is taken at stage α, then $\mathbb{P}_\alpha$ is κ-directed closed.*

3. *If $\mathbb{P}_\alpha$ is a forcing iteration of $\langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ where all limits are direct or inverse, and for each $\beta < \alpha, \dot{\mathbb{Q}}_\beta$ is κ-directed closed in $V^{\mathbb{P}_\beta}$, then if inverse limits are taken at all stages β such that $\mathrm{cf}(\beta) \leq \kappa$, then $\mathbb{P}_\alpha$ is κ-directed closed.*

Given an α-iteration $\mathbb{P}_\alpha$ and an ordinal $\gamma < \alpha$, it is often useful to decompose $\mathbb{P}_\alpha$ into forcing notions $\mathbb{P}_\gamma * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is a suitable notion of forcing in $V^{\mathbb{P}_\gamma}$ which represents the iteration from stage γ to stage α. The following lemma makes this idea precise.

If $\beta < \alpha$ and $p \in \mathbb{P}_\alpha$, let $p^{(\beta)} = p \restriction \{\gamma : \beta \leq \gamma < \alpha\}$, so that $p = (p \restriction \beta) \cup p^{(\beta)}$. Let $\mathbb{P}_{\beta,\alpha} = \{p^{(\beta)} : p \in \mathbb{P}_\alpha\}$. Given G_β which is $\mathbb{P}_\beta$-generic, define an ordering on $\mathbb{P}_{\beta,\alpha}$ by setting

$$f \leq_{\mathbb{P}_{\beta,\alpha}} g \text{ if and only if } (\exists p \in G_\beta) p \cup f \leq p \cup g.$$

Let $\dot{\mathbb{P}}_{\beta,\alpha}$ be a term of the forcing language with $\mathbb{P}_\beta$ which denotes $(\mathbb{P}_{\beta,\alpha}, \leq_{\mathbb{P}_{\beta,\alpha}})$.

Lemma 1.2.11. $\mathbb{P}_\alpha$ *is isomorphic to a dense subset of $\mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta,\alpha}$. Moreover,*

$$\Vdash_{\mathbb{P}_\beta} \text{``}\dot{\mathbb{P}}_{\beta,\alpha} \text{ is isomorphic to an } (\alpha - \beta)\text{-iteration.''}$$

Proof. See [1]. □

The reader should note that there are subtleties to worry about with respect to terms in different languages, but these correspondences are canonical, and so we will not worry about the details here.

1.3   LARGE CARDINALS

In this section, we give some preliminary results on the large cardinals we will use in this thesis. Our treatment loosely follows [5], but any other treatment of this material should not be too dissimilar from this.

First, we recall the ultrapower construction. Let $\mathcal{U}$ be a filter on a set X. Then U is an *ultrafilter* if it is a maximal filter with respect to set inclusion, κ-*complete* if it is closed under intersections of length less than κ, and *non-principal* if it contains no singletons. If M is a transitive set model of ZFC and U is an ultrafilter on a set X, define an equivalence relation $\sim_U$ on M^X by letting $f \sim_U g$ iff $\{x : f(x) = g(x)\} \in U$. Define the model $\mathrm{Ult}(M, U)$ to be the structure with universe $\left(\prod_{x \in X} M\right) / \sim_U$ and a relation symbol $\in_U$ where $f \in_U g$ iff $\{x : f(x) \in g(x)\} \in U$. The following fundamental theorem highlights the importance of ultraproducts.

Theorem 1.3.1 (Łoś)**.** *Let U be an ultrafilter on a set X and let $\mathrm{Ult}(M, U)$ be as above. Then for every n-ary formula φ in the language of set theory, we have*

$$\mathrm{Ult}(M, U) \vDash \varphi([f_1], ..., [f_n]) \iff \{x \in X : M \vDash \varphi(f_1(x), ..., f_n(x))\} \in U.$$

Proof. By induction on φ. □

Let M, N be transitive models of set theory. We say a function $j : M \to N$ is an *elementary embedding* if for every formula φ in the language of set theory and every $x_1, ..., x_n \in M$, we have

$$M \vDash \varphi(x_1, ..., x_n) \iff N \vDash \varphi(j(x_1), ..., j(x_n)).$$

If $x \in M$, let c_x denote the constant function on X with value x. By Łoś' theorem, the map $j : M \to \mathrm{Ult}(M, U)$ given by $j(x) = [c_x]$ is actually an elementary embedding.

The following is an easy lemma regarding elementary embeddings.

Lemma 1.3.2. Suppose $j : M \to N$ is an elementary embedding between transitive models of set theory which is not the identity function. Then,

1. There is an ordinal α such that $j(\alpha) \neq \alpha$.

2. j is non-decreasing on the ordinals of M.

Proof. For the first claim, suppose α is the least ordinal such that for some set x with rank α, $j(x) \neq x$. We wish to show that $j(\alpha) \neq \alpha$. To the contrary, suppose $j(\alpha) = \alpha$, and notice by elementarity that $\mathrm{rank}(j(x)) = \alpha$. Then, for all $z \in x \cup j(x)$, $j(z) = z$. Again, by elementarity, $z \in x \iff j(z) \in j(x)$. So, $j(x) = x$, a contradiction.

For the second claim, we prove by induction on α that $j(\alpha) \geq \alpha$. Suppose $\beta + 1 = \alpha$ and $j(\beta) \geq \beta$. By elementarity, $j(\beta + 1) = j(\beta) + 1$, so $j(\beta + 1) \geq \beta + 1$. If α is limit and $j(\gamma) \geq \gamma$ for every $\gamma < \alpha$, then by elementarity, $j(\gamma) < j(\alpha)$, so $j(\alpha) \geq \alpha$. $\square$

In light of this lemma, we call the least ordinal α so that $j(\alpha) > \alpha$ the *critical point* of j, denoted $\mathrm{crit}(j)$.

Recall that we can extend this idea to form ultrapowers of the universe V, even though it is a proper class, via *Scott's trick*: If f is a function with domain X, let

$$[f] = \left\{ g : \begin{array}{l} (g \text{ is a function on } X) \text{ and} \\ (g \sim_U f) \text{ and} \\ \forall h[(h \text{ is a function on } X \ \wedge \ h \sim_U f \implies \mathrm{rank}(h) \geq \mathrm{rank}(g))] \end{array} \right\}$$

and say that $[f] \in_U [g]$ iff $\{x \in X : f(x) \in g(x)\} \in U$. Notice that with these modifications, we have that for all functions f with domain X, $\{[g] : [g] \in_U [f]\}$ is a

18

set, so let $\mathrm{Ult}(V, U)$ be the structure with universe $\{[f] : f$ is a function on $X\}$ and relation symbol $\in_U$. The statement of Łoś's theorem goes through just like before.

The following lemma shows that the ultrapower is well-founded when the ultrafilter in question is sufficiently closed.

Lemma 1.3.3. Suppose U is an ω_1-complete ultrafilter on a set X. Then $\mathrm{Ult}(V, U)$ is well-founded.

Proof. Suppose $\{[f_n] : n < \omega\}$ is a sequence of elements in $\mathrm{Ult}(M, U)$ so that $f_{n+1} \in_U [f_n]$ for each n. By Łoś's theorem, for each n, the set $A_n = \{x \in X : f_{n+1}(x) \in f(x)\}$ is in U. By countable completeness, $A := \bigcap_{n<\omega} A_n \in U$. But if $x \in A$, then $\{f_n(x) : n < \omega\}$ forms an infinite descending sequence in V, a contradiction. $\qquad\square$

By Mostowski's collapse theorem, every well-founded model is isomorphic to a transitive set-like model, and so from this point forward, we identify ultrapowers of the universe with their transitive collapses.

Recall the equivalent formulations of a measurable cardinal. We give a sketch here for completeness' sake.

Theorem 1.3.4. *Let κ be an uncountable cardinal. Then the following are equivalent:*

1. *There is an inner model M of ZFC and a nontrivial elementary embedding $j : V \to M$ with critical point κ.*

2. *There is a κ-complete nonprincipal ultrafilter on κ.*

Proof. Suppose M is an inner model of ZFC and $j : V \to M$ is a nontrivial elementary embedding with $\kappa = \mathrm{crit}(j)$, which, by elementarity, is uncountable. Then let $U = \{X \subseteq \kappa : \kappa \in j(X)\}$. It is easy to see that U is a nonprincipal ultrafilter by elementarity. It remains to show that U is κ-complete. Suppose $\alpha < \kappa$ and suppose $\vec{X} = \langle X_\gamma : \gamma < \alpha \rangle$ is an α-sequence of elements of U. By elementarity, $j(\vec{X})$ is a

sequence of length $j(\alpha)$ of subsets of $j(\kappa)$, and since κ is the critical point of j, we have that $j(\vec{X}) = \langle j(X_\gamma) : \gamma < \alpha \rangle$. Thus, $j(\bigcap_\gamma X_\gamma) = \bigcap_\gamma j(X_\gamma)$, and so $\bigcap_\gamma X_\gamma \in U$.

For the converse, suppose U is a κ-complete ultrafilter on κ and let $M = \mathrm{Ult}(V, U)$. Then let $j : V \to M$ be the embedding

$$x \mapsto [c_x],$$

where c_x is the constant function with value x. It remains to prove that $\mathrm{crit}(j) = \kappa$. It can be shown by induction on α that $j(\alpha) = \alpha$ for all $\alpha < \kappa$, and it can also be shown that if d is the identity function on κ, then $[d] \geq \kappa$ and $[d] < j(\kappa)$, the details of which are omitted.

$\square$

A cardinal κ as in Theorem 1.3.4 is called a *measurable cardinal*.

Remark 1.3.5. For the remainder of this thesis, all measures on measurable cardinals are assumed to be *normal*, i.e. closed under diagonal intersections of length $< \kappa$ (equivalently, $[\mathrm{id}] = \kappa$), unless otherwise specified. It is well-known that every measurable cardinal carries a normal measure, so the distinction is not important here.

Furthermore, we will implicitly assume that all ultrafilters are nonprincipal, and we will not state that an ultrafilter on a measurable cardinal κ is κ-complete. In short, we say "U is a measure on κ" or U is an ultrafilter on κ" instead of "U is a nonprincipal κ-complete normal ultrafilter on κ."

Stronger large cardinal axioms can be defined by imposing more closure on the target model M. This can be done in a variety of ways to define a variety of large cardinal axioms, but the only one we will use in this thesis is *supercompactness*, defined below.

Definition 1.3.6. A cardinal κ is λ-*supercompact* if there is a nontrivial elementary

embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $^\lambda M \subseteq M$. A cardinal κ is *supercompact* if it is λ-supercompact for every λ.

Just as with measurability, there is an equivalent definition of λ-supercompactness in terms of ultrafilters, but we will not need this formulation.

1.4 EXTENDERS

This section gives an introduction to the theory of extenders. Extenders play a central role in modern set theory, and one could easily spend a semester exploring just the basics. We only record here the information which we will need in Chapter 2: in particular we will not use extenders for any inner model theory, which greatly simplifies things. Our treatment loosely follows that of [6].

Definition 1.4.1. Let $j : V \to M$ be an elementary embedding with critical point κ and let $\kappa \leq \lambda \leq j(\kappa)$. For every finite subset $a \subseteq \lambda$, let $E_a = \{X \subseteq [\kappa]^{<\omega} : a \in [\lambda]^{<\omega}\}$. The (κ, λ)-*extender derived from j* is the set $E = \{E_a : a \in [\lambda]^{<\omega}\}$.

Lemma 1.4.2. For each $a \in [\lambda]^{<\omega}$, E_a is a κ-complete measure over $[\kappa]^{|a|}$ with critical point κ which is principal iff $a \subseteq \kappa$.

Proof. Since $\lambda \leq j(\kappa)$ we get that $a \in [\lambda]^{|a|} \subseteq [j(\kappa)]^{|a|} = j\left([\kappa]^{|a|}\right)$, so $[\kappa]^{|a|} \in E_a$. By elementarity, E_a is a filter. If $a \notin j(x)$ then $a \in j(x)^c = j(x^c)$, so E_a is also an ultrafilter. As the critical point is κ, j also preserves intersections of $< \kappa$ many sets, making E_a κ-complete.

Finally, assume that E_a is principal, so that $\{x\} \in E_a$ for some $x \in [\kappa]^{|a|}$. Then $a \in j(\{x\}) = \{j(x)\}$, meaning $a = j(x)$. But x is below the critical point, so $j(x) = x$, meaning $a \subseteq \kappa$. Conversely if $a \subseteq \kappa$, then we show that $\bigcap E_a \in E_a$. As a is below the critical point, $a = j(a)$, implying that $j(a) = a \in j\left(\bigcap E_a\right)$ holds iff $a \in \bigcap E_a$ holds, of which the latter is true since $a \in j(x) \cap [\kappa]^{<\omega} = x$ for every $x \in E_a$. $\square$

Suppose $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$. Suppose $b = \{\alpha_1, ..., \alpha_n\}$ where $\alpha_1 < ... < \alpha_n$ and $a = \{\alpha_{i_1}, ..., \alpha_{i_m}\}$. Then if $\pi_{b,a} : [\lambda]^{|b|} \to [\lambda]^{|a|}$ is defined by

$$\pi_{b,a}(\{\xi_1, ..., \xi_n\}) = \{\xi_{i_1}, ..., \xi_{i_m}\},$$

then it can be easily shown that

$$X \in E_a \iff \{t : \pi_{b,a}(t) \in X\} \in E_b.$$

Thus, if we define

$$i_{a,b} : \mathrm{Ult}(V, E_a) \to \mathrm{Ult}(V, E_b); \quad [f]_{E_a} \mapsto [f \circ \pi_{b,a}]_{E_b},$$

then

$$\langle (\mathrm{Ult}(V, E_a), i_{a,b}) : a, b \in [\lambda]^{<\omega}, \ a \subseteq b \rangle$$

forms a directed system of elementary embeddings, and we may form its direct limit.

Definition 1.4.3. The *ultrapower of a (κ, λ)-extender* E is given by

$$\mathrm{Ult}(V, E) := \varinjlim_{a \in [\lambda]^{<\omega}} \mathrm{Ult}(V, E_a),$$

where the direct limit is over the system $\{i_{a,b}\}$ defined above.

We now analyze the embedding $j_E : V \to \mathrm{Ult}(V, E)$ more closely. Suppose $a, b \in [\lambda]^{<\omega}$. It is simple to check that if $k_a : \mathrm{Ult}(V, E_a) \to M$ is defined by

$$k_a([f]_{E_a}) = j(f)(a)$$

and $j_a : V \to \mathrm{Ult}(V, E_a)$ is the usual ultrapower map, then k_a is elementary and $k_a \circ j_a = j$. Now define

$$k_{aE} : \mathrm{Ult}(V, E_a) \to \mathrm{Ult}(V, E), \text{ and}$$

$$k_E : \mathrm{Ult}(V, E) \to M$$

22

so that

$$k_E \circ j_E = j, \ k_{aE} \circ j_a = j_E, \text{ and } k_E \circ k_{aE} = k_a.$$

We have the following lemma, with an alternative characterization of the model $\mathrm{Ult}(V, E)$.

Lemma 1.4.4. $\mathrm{Ult}(V, E) = \{j_E(f)(a) : a \in [\lambda]^{<\omega}, \ f \in V, \ \mathrm{dom}(f) = [\kappa]^{|a|}, \ a \in \mathrm{dom}(f)\}$.

Proof. If $x \in \mathrm{Ult}(V, E)$, then there is an $a \in [\lambda]^{<\omega}$ and an $f \in V$ such that $x = k_{aE}([f]_{E_a})$, so

$$k_E(x) = k_E(k_{aE}([f]_{E_a})) = k_a([f]_{E_a}) = j(f)(a).$$

Thus,

$$\mathrm{ran}(k_E) = \{j_E(f)(a) : a \in [\lambda]^{<\omega}, \ f \in V, \ a \in \mathrm{dom}(f)\}.$$

But notice that since $k_E \circ j_E = j$, we also have $M_E = \mathrm{ran}(k_E)$, completing the proof of the lemma. $\square$

1.5   SHARPS AND DAGGERS

In this section, we review some of the basic notions surrounding sharps. We assume that the reader is relatively familiar with the basic theory of $0^\sharp$, but is not familiar with its role in inner model theory. We first state several equivalent formulations of the existence of $0^\sharp$. Our treatment loosely follows that of [13].

Recall two equivalent characterizations of the existence of $0^\sharp$.

Theorem 1.5.1. *The following are equivalent:*

1. *There exists a nontrivial elementary embedding $j : L \to L$.*

2. *There is a proper class of indiscernibles for L.*

Proof. See [5] or [6]. $\square$

We say that "$0^\sharp$ exists" if either clause of Theorem 1.5.1 is true, and define

$$0^\sharp = \{\ulcorner\varphi\urcorner : \varphi^L(c_0, ..., c_{n-1})\},$$

where $c_0, ..., c_{n-1}$ are indiscernibles for L and $\ulcorner\varphi\urcorner$ denotes the Gödel number of φ.

We now isolate the particular consequences of the existence of $0^\sharp$ which we will use much later. Suppose there is a nontrivial elementary embedding $j : L \to L$ with critical point κ. As with elementary embeddings with full measures on measurable cardinals, we may define an ultrafilter κ by setting

$$U_j = \{X \subseteq \kappa : X \in L \text{ and } \kappa \in j(X)\}.$$

As expected, we have that

Lemma 1.5.2. Suppose that $j : L \to L$ is an elementary embedding with critical point κ. Then

1. U_j is a nonprincipal ultrafilter on $L \cap \mathcal{P}(\kappa)$

2. U_j is L-normal, i.e. U_j satisfies Fodor's lemma for regressive functions in L, and

3. U_j is L-κ-complete, i.e. U_j is closed under sequences in L of length $< \kappa$ whose entries are members of L

Proof. Easy. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 1.5.3. The reader should be careful and note that the ultrapower $\mathrm{Ult}(L, U_j)$ is different from the ultrapowers defined previously since U_j only contains subsets of κ which are in L. These ultrapowers are often called Σ_0-ultrapowers, since the map i is a Σ_0-elementary embedding, and the usual proof of Łoś' Theorem works for Σ_0-formulas. In this particular case, i is actually fully elementary and Łoś' Theorem holds for all formulas, but this is not in general true: When taking Σ_0-ultrapowers of the form $\langle L, \in, U_j\rangle$, it will not necessarily be true that Łoś' Theorem is true for more than just Σ_0-formulas in the expanded language.

One may ask the question of how far up the L-hierarchy one must go to see that U_j is an *amenable* ultrafilter, and the answer is as follows:

Lemma 1.5.4. Let $\lambda = (\kappa^+)^L$. Then

1. U_j is an ultrafilter on $L_\lambda \cap \mathcal{P}(\kappa)$,

2. U_j is L_λ-normal,

3. U_j is L_λ-κ-complete, and

4. U_j is L_λ-amenable.

Proof. Parts 1-3 are the same as Lemma 1.5.2. For the fourth part, we wish to show that if $\alpha < \lambda$, then $U_j \cap L_\alpha \in L_\lambda$. If $\alpha < \lambda$, then $|L_\alpha|^L = |H_\alpha||\kappa$. Let $\langle A_\eta : \eta < \kappa \rangle$ be an enumeration in L of $\mathcal{P}(\kappa) \cap L_\alpha$. Then if

$$j(\langle A_\eta : \eta < \kappa \rangle),$$

then

$$U_j = \{B_\eta \cap \kappa : \eta < \kappa \text{ and } \kappa \in B_\eta\}.$$

$\square$

The formulation of $0^\sharp$ for L can be relativized in a straightforward way to inner models $L[X]$ for sets $X \subseteq \mathrm{On}$ to produce corresponding sets $X^\sharp \subseteq \mathrm{On}$.

Theorem 1.5.5. *The following are equivalent:*

1. *There exists a nontrivial elementary embedding $j : L[X] \to L[X]$.*

2. *There is a proper class of indiscernibles for $L[X]$.*

Proof. The proof follows similarly to the proof of Theorem 1.5.1. Once again, see [5] or [6] for the details. $\square$

If X is a set of ordinals, then we say that "$X^\sharp$ exists" if and only if either clause of Theorem 1.5.5 holds, and define

$$X^\sharp = \{\ulcorner\varphi\urcorner : \varphi^{L[X]}(c_0, ..., c_{n-1})\},$$

where $c_0, ..., c_{n-1}$ are indiscernibles for $L[X]$.

One can run the same procedure further up the large cardinal hierarchy: In order to construct the sharp of a large cardinal property, we start with a 'minimal' inner model M for the large cardinal property,[1] assuming that there is a closed proper class I of indiscernibles for M, and defining a new real, just as with $0^\sharp$, to be the set of Gödel numbers of formulas $\varphi(c_0, ..., c_{n-1})$ such that $M \vDash \varphi(c_0, ..., c_{n-1})$ for any $(c_0, ..., c_{n-1}) \in [I]^n$. For example, for the existence of one measurable cardinal, we can consider the model $M = L[U]$ such that U is a normal ultrafilter in M. If I is a closed proper class of indiscernibles for M, then we can define a new real, $0^\dagger$, as

$$0^\dagger = \{\ulcorner\varphi\urcorner : \varphi^{L[U]}(c_0, ..., c_{n-1})\},$$

where $(c_0, ..., c_{n-1}) \in I^n$. It can be shown that this process yields a unique real even though the model M may not be unique, as in the case of $L[U]$.

This procedure may be continued further up the large hierarchy, yielding sharps for the existence of many measurable cardinals, strong cardinals, Woodin cardinals, and more. Viewing sharps from an inner model theoretic perspective will be explored further in Chapter 3.

---

[1]The word 'minimal' here is being used informally and is not well-defined. Nonetheless, this still serves as a useful intuitive explanation.

CHAPTER 2

THE UPPER BOUND

With preliminaries taken care of, we may now begin to prove the desired equiconsistency, and this chapter aims to prove the upper bound. In order to do so, we must prove that would like to prove that the consistency of the theory

$$\text{ZFC} + \text{``there exists a measurable cardinal } \kappa \text{ with } o(\kappa) = \kappa^{++}\text{''}$$

implies the consistency of the theory

$$\text{ZFC} + \text{``there exists a measurable cardinal } \lambda \text{ with } 2^\lambda > \lambda^+\text{.''}$$

Section 2.1 details an proof due to Silver of this consistency from much stronger hypotheses than $o(\lambda) = \lambda^{++}$ for expository purposes. Section 2.2 extrapolates some of the key ideas in Silver's proof to serve as a catalogue of lemmas to refer to later. Section 2.3 gives the upper bound from optimal hypotheses, building on the ideas from the previous two sections.

2.1   SILVER'S PROOF

As just mentioned, we will now outline a proof originally due to Silver that the GCH can fail at a measurable cardinal from very strong hypotheses (the existence of a cardinal κ which is κ^{++}-supercompact). Since the GCH failing at a measurable cardinal implies the GCH failing on a measure-one set of smaller cardinals, one natural way to producing a failure of GCH at a measurable cardinal is to start with a model of GCH with a measurable cardinal κ and perform an Easton support iteration violating GCH on $X \cup \{\kappa\}$ for some measure-one $X \subseteq \kappa$.

Theorem 2.1.1. *If there exists a cardinal κ which is κ^{++}-supercompact, then there is a generic extension in which κ is measurable and $2^\kappa > \kappa^+$.*

Proof. We may assume that the GCH holds in V without loss of generality, as otherwise we may perform a preliminary forcing to force the GCH. Suppose κ is a κ^{++}-supercompact cardinal, witnessed by an elementary embedding $j : V \to M$, and let X be any set of inaccessible cardinals so that $\kappa \in j(X)$. We will define an Easton support iteration which violates the GCH at κ and at every cardinal in X. We inductively define the Easton iteration $\mathbb{P}_\alpha$ as follows:

If $\alpha \in X \cup \{\kappa\}$, let $\dot{\mathbb{Q}}_\alpha = (\mathrm{Add}(\alpha, \alpha^{++}))^{V^{\mathbb{P}_\alpha}}$, and let $\dot{\mathbb{Q}}_\alpha = \{1\}$ otherwise. Let $\mathbb{P} = \mathbb{P}_{\kappa+1}$.

Let G be a generic filter on $\mathbb{P}$. We wish to prove that κ is measurable in $V[G]$, and that $V[G] \vDash 2^\kappa = \kappa^{++}$. First, we show that $V[G] \vDash$ "$2^\kappa = \kappa^{++}$ and κ is a regular cardinal".

First, note that $\mathbb{P}$ is isomorphic to the two-step iteration $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$, so that if G_κ is $\mathbb{P}_\kappa$-generic over V and H_κ is $\mathbb{Q}_\kappa = i_{G_\kappa}(\dot{\mathbb{Q}}_\kappa)$-generic over $V[G_\kappa]$, then $V[G] = V[G_\kappa][H_\kappa]$. Notice that a direct limit is taken at $\mathbb{P}_\kappa$, and since there is a stationary set of inaccessible cardinals below each measurable cardinal, there is a stationary set of $\alpha < \kappa$ where a direct limit is taken at $\mathbb{P}_\alpha$. By Theorem 1.2.9, $\mathbb{P}_\kappa$ has the κ-chain condition. Thus, by Theorem 1.1.9, κ is regular in $V[G_\kappa]$. Furthermore, $\mathbb{Q}_\kappa$ preserves κ's regularity because it preserves all cardinals. So, κ is regular in $V[G_\kappa][H_\kappa] = V[G]$. It is clear by construction that $2^\kappa = \kappa^{++}$ in $V[G]$.

All that needs to be shown is that κ is measurable in $V[G]$. Consider the model $M[G]$.

Lemma 2.1.2.

$$(M[G])^{\kappa^{++}} \cap V[G] \subseteq M[G].$$

Proof. Let $f : \kappa^{++} \to \mathrm{Ord}$ be a function in $V[G]$. We wish to show that it also is

in $M[G]$. Let $\dot{f}$ be a name for f, and let p be a condition which forces that $\dot{f}$ is a function from κ^{++} to the ordinals. For each $\alpha < \kappa^{++}$ and each $q \leq p$ such that there is a β such that $p \Vdash \dot{f}(\alpha) = \beta$, define a function h so that $h(\alpha, q)$ is the unique β such that $p \Vdash \dot{f}(\alpha) = \beta$. Clearly, $h \in M$ because h has size κ^{++}, and so f can be defined in $M[G]$ as being the function which maps α to the unique β such that for some $q \in G$, $h(\alpha, q) = \beta$. $\qquad\square$

We now analyze $j(\mathbb{P})$. The key fact is Lemma 2.1.3, which says that $j(\mathbb{P})$ is simply $\mathbb{P}$ augmented by a κ^{++}-directed closed notion of forcing.

Lemma 2.1.3. $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{S}}$ for some notion of forcing $\dot{\mathbb{S}} \in M^{\mathbb{P}}$ such that $M^{\mathbb{P}} \vDash$ "$\dot{\mathbb{S}}$ is a κ^{+++}-directed closed notion of forcing."

Proof. In M, $j(\mathbb{P})$ is a notion of forcing obtained by iterating up to $j(\kappa) + 1$. We claim that we can apply Lemma 1.2.11 to $j(\mathbb{P})$ at $\kappa + 1$. First, note that $(j(\mathbb{P}))_\alpha = \mathbb{P}_\alpha$ for every $\alpha < \kappa$ since $\mathrm{crit}(j) = \kappa$. Since a direct limit is taken at stage κ, we also have that $(j(\mathbb{P}))_\kappa = \mathbb{P}_\kappa$. Since $(\dot{\mathbb{Q}}_\kappa)^V = (\dot{\mathbb{Q}}_\kappa)^M$, we also have that $(j(\mathbb{P}))_{\kappa+1} = \mathbb{P}_{\kappa+1}$. The first nontrivial step above κ in the iteration occurs at the least inaccessible above κ, and thus the first nontrivial direct limit is taken far above κ^{++}, and then only at regular cardinals. Since $|\mathbb{P}_{\kappa+1}|$, the hypotheses of Lemma 1.2.11 are satisfied.

Hence we may factor $j(\mathbb{P})$ at level $\kappa + 1$ to see that

$$j(\mathbb{P}) \cong (j(\mathbb{P}))_{\kappa+1} * (j(\mathbb{P}))_{\kappa+1, j(\kappa)+1}.$$

We just argued that the first factor equals $\mathbb{P}$. Denote by $\dot{\mathbb{S}}$ the second factor. Note that $\dot{\mathbb{S}}$ is, in $M^{\mathbb{P}}$, a notion of forcing obtained by iterating, with Easton support, from $\kappa + 1$ to $j(\kappa) + 1$. Every stage in this iteration is κ^{+++}-directed closed because every stage is either the trivial notion of forcing or Cohen forcing. Because we are using an Easton support iteration, we may use Lemma 1.2.10 to see that $M^{\mathbb{P}} \vDash$ "$\dot{\mathbb{S}}$ is a κ^{+++}-directed closed notion of forcing." This completes the proof of Lemma 2.1.3. $\quad\square$

By Lemma 2.1.3, if $p \in \mathbb{P}$, then $j(p) = (p', \dot{s})$ where $p' \in \mathbb{P}$ and $\dot{s} \in \dot{\mathbb{S}}$. By the definition of $\mathbb{P}$, $p = \langle p_\xi : \xi < \kappa + 1 \rangle$ and there is $\xi_0 < \kappa$ such that $p_\xi = 1$ for all ξ such that $\xi_0 \leq \xi < \kappa$. Thus $j(p) = \langle p'_\xi : \xi < j(\kappa) + 1 \rangle$ and $p'_\xi = 1$ for all ξ, $\xi_0 \leq \xi < j(\kappa)$. In particular, $p'_\kappa = 1$, and since $p'_\xi = p_\xi$ for all $\xi < \kappa$, and $p = j(p) \restriction (\kappa + 1)$, we have that $p' = (p \restriction \kappa)^\frown 1$. This implies that if $p \in G$ and $j(p) = (p', \dot{r})$, then $p' \in G$.

Let $D = \{ r \in i_G(\dot{\mathbb{S}}) : \exists p \in G, q = i_G(\dot{s}) \text{ where } j(p) = (p', \dot{s}) \}$. Since $\mathbb{P}$ has size κ^{++}, we have that $j \restriction \mathbb{P} \in M$, and therefore $D \in M[G]$. D is directed because G is directed. In $M[G]$, we have $|D| \leq |G| \leq |\mathbb{P}| = \kappa^{++}$, and since $i_G(\dot{\mathbb{S}})$ is κ^{+++}-directed closed, there exists some $a \in i_G(\dot{\mathbb{S}})$ so that $a \leq r$ for all $r \in D$.

We now consider a generic extension of $V[G]$. Let H be a $V[G]$-generic filter on $i_G(\dot{\mathbb{S}}$ which contains a. Since H is also $M[G]$-generic and $\mathbb{P}$ is an initial segment of $j(\mathbb{P})$, there is an I which is $j(\mathbb{P})$-generic over M with $M[I] = M[G][H]$.

Now we would like to extend the embedding $j : V \to M$ to an embedding $j^+ : V[G] \to M[I]$ defined in $V[G][H]$ For each $x \in V[G]$, let $j^+(x) = i_I(j(\dot{x}))$, where $\dot{x}$ is a $\mathbb{P}$-name for x. (Notice that "i_I" is not an elementary embedding, but rather interpretation by the generic I.) It is clear that $j^+ \restriction V = j$.

We need to show that j^+ is well-defined. Suppose $p \in G$ forces that $\dot{x} = \dot{y}$. We would like to show that $j^+(p) \in I$ forces that $j^+(\dot{x}) = j^+(\dot{y})$. It is enough to show that $j^+ "G \subseteq I$. Here is where we use the condition a. If $p \in G$, then as before, $j^+(p) = (p', r)$. We have shown previously that $p' \in G$, and since $p \in G$, we have that $r \geq a$, and so $r \in H$ Thus, by the definition of I, we have $(p', r) \in I$, which shows that $j^+ "G \subseteq I$, and hence j^+ is well-defined.

So, we have an elementary embedding $j^+ : V[G] \to M[I]$, and, in $V[G]$, we can define an ultrafilter U on κ in the usual way by letting

$$U = \{ X \subseteq \kappa : \kappa \in j^+(X) \}.$$

We need to show that $U \in V[G]$. We have already shown that $i_G(\dot{\mathbb{S}})$ is κ^{++}-closed in $V[G]$, so $V[G][H]$ does not have any κ^{++}-sequences not in $V[G]$. Since $|U| = \kappa^{++}$,

we have that $U \in V[G]$. It is clear that U is nonprincipal and κ-complete. Thus,

$$V[G] \vDash \text{``}\kappa \text{ is a measurable cardinal''},$$

which completes the proof of Theorem 2.1.1.

$\square$

2.2 FORCING AND ELEMENTARY EMBEDDINGS

This section documents some of the key features of Silver's proof which we will use extensively in the next section, and serves as a catalogue of lemmas to refer to in the next section. Most of these proofs are originally folklore, but our organization of the results loosely follows [2].

First, notice that in the previous proof, we were able to lift the embedding $j :$ $V \rightarrow M$ to an embedding $j^+ : V[G] \rightarrow M[I]$. This is a very useful technique in general, and we prove for completeness sake that this can always be done under the same conditions that were satisfied in the previous theorem.

Lemma 2.2.1. Let $j : M \rightarrow N$ be an elementary embedding between transitive models of set theory. Suppose $\mathbb{P}$ is a notion of forcing with $\mathbb{P} \in M$, G is $\mathbb{P}$-generic over M, and H is $j(\mathbb{P})$-generic over N. If $j``G \subseteq H$, then there exists an elementary embedding $j^+ : M[G] \rightarrow N[H]$ extending j with $j^+(G) = H$.

Proof. Suppose $j``G \subseteq H$, and let $j^+ : M[G] \rightarrow N[H]$ be the map

$$j^+(i_G(\dot{\tau})) = i_H(j(\dot{\tau})).$$

First, we show j^+ is well-defined. Suppose τ_1 and τ_2 are $\mathbb{P}$-names for the same set in $M[G]$, so that $i_G(\dot{\tau_1}) = i_G(\dot{\tau_2})$. By Theorem 1.1.1, there is a $p \in G$ such that $p \Vdash_{\mathbb{P},M} \dot{\tau_1} = \dot{\tau_2}$. By the elementarity of j, $j(p) \Vdash_{j(\mathbb{P}),N} i_H(j(\dot{\tau_1})) = i_H(j(\dot{\tau_2}))$.

The same argument shows that j^+ is elementary.

Next, we show that j^+ extends j. Suppose $x \in M$ and $\check{x}$ is the canonical $\mathbb{P}$-name for x. Since j is elementary, $j(\check{x})$ is the canonical $j(\mathbb{P})$-name for $j(x)$, and so

$$j^+(x) = j^+(i_G(\check{x})) = i_H(k(\check{x})) = j(x).$$

Finally, we show that $j^+(G) = H$. Let Γ denote the canonical $\mathbb{P}$-name for $\mathbb{P}$. By elementarity, $j(\Gamma)$ is the canonical $j(\mathbb{P})$-name for the $j(\mathbb{P})$-generic filter. Thus, $j(G) = H$. $\qquad\square$

A central problem in the theory of forcing and elementary embeddings is coming up with clever ways to satisfy the hypotheses of Lemma 2.2.1, i.e. that $j``G \subseteq H$. The following Lemma gives one way this can be done, namely by transferring a generic filter through an elementary embedding.

Lemma 2.2.2. Let $j : M \to N$ be an elementary embedding, let $\mathbb{P} \in M$ be a separative notion of forcing, and suppose

$$M \vDash \text{``}\mathbb{P} \text{ is } \lambda^+\text{-distributive.''}$$

Suppose further that every element of N can be written in the form $j(f)(a)$ for some $a \in N$ and some $f \in M$ such that $M \vDash |\mathrm{dom}(f)| \leq \lambda$. Let G be $\mathbb{P}$-generic over M and let H be the filter on $j(\mathbb{P})$ generated by $j``G$. Then H is $j(\mathbb{P})$-generic over N.

Proof. Suppose D is a dense open subset of $j(\mathbb{P})$. By our hypothesis on N, we may write D in the form $j(f)(a)$ for some $a \in N$ and some $f \in M$ such that $M \vDash |\mathrm{dom}(f)| \leq \lambda$. Without loss of generality, assume that $f(X)$ is a dense open subset of $\mathbb{P}$ for every $X \in \mathrm{dom}(f)$.

Let $E = \bigcap_{X \in \mathrm{dom}(f)} f(X)$. By our hypothesis on X, E is an intersection of fewer than λ^+-many dense open sets, so by the λ^+-distributivity of $\mathbb{P}$, E is a dense subset of $\mathbb{P}$. Thus, $E \cap G \neq \emptyset$, so pick a $p \in E \cap G$. By the elementarity of j and the definition of E, we have $k(p) \in k(f)(a)$. Thus, $k(p) \in D$, so $H \cap D \neq \emptyset$, and so H is $j(\mathbb{P})$-generic over N. $\qquad\square$

It is important to note here that the the conclusion of this lemma, by Lemma 2.2.1, allows us to extend j to an elementary embedding $j^+ : M[G] \to N[H]$ with $j^+(G) = H$.

Lemma 2.2.3 is just Lemma 2.1.2 in a general setting.

Lemma 2.2.3. Let M and N be inner models of ZFC with $M \subseteq N$ and let $\mathbb{P} \in M$ be a notion of forcing. Then if $N \vDash {}^{<\kappa}M \subseteq M$, $N \vDash$ "$\mathbb{P}$ is κ-c.c." and G is $\mathbb{P}$-generic over N, then $N[G] \vDash {}^{<\kappa}M[G] \subseteq M[G]$.

Proof sketch. Suppose a condition p forces that f is a bijection from some $\mu < \lambda$ and taking values in $M[G]$. Since $\mathbb{P}$ is λ-c.c. in N, $N \vDash$ " for every $\alpha < \mu$, there are $< \lambda$ many possible values for $f(\alpha)$. So f can be written as an M-name since ${}^{<\lambda}M \subseteq M$. $\qquad\square$

The last lemma is a generalization of the Rasiowa-Sikorski lemma. While it is not an obvious streamlining of a part of Silver's proof, it will prove to be useful to cite as the optimal upper bound proof gets more sophisticated.

Lemma 2.2.4. Let M and N be inner models with $M \subseteq N$ and let $\mathbb{P} \in M$ be a notion of forcing.

Let λ be a cardinal in N, and suppose

$N \vDash$ "$\mathbb{P}$ is λ-strategically closed and there are at most λ-many antichains of $\mathbb{P}$ in M."

Then for any condition $p \in \mathbb{P}$, N contains exactly 2^λ-many filters containing p that are $\mathbb{P}$-generic over M.

Proof. Work in N. Let $\langle A_\alpha : \alpha < \lambda \rangle$ enumerate the antichains of $\mathbb{P}$ in M, and let τ be a winning strategy for player I in the λ-strategic closure game for $\mathbb{P}/p$. Now we build a tree of conditions $T = \langle p_t : t \in {}^{<\lambda}2 \rangle$ of conditions in $\mathbb{P}/p$ with the goal of having each branch of T generate a unique generic filter over $\mathbb{P}$ containing p. Let

1. $p_{\langle\rangle} = p$

2. If length(t) is even, say $2 \cdot \alpha$, then let p_t be so that $p_{t^\frown 0} \perp p_{t^\frown 1}$ and so that $p_{t^\frown 0}$ and $p_{t^\frown 1}$ both refine some element of A_α.

3. If length(t) $= 2 \cdot \alpha + 1$, then p_t is the response dictated by τ at move $2 \cdot \alpha$ in the λ-strategic closure game for $\mathbb{P}/p$, where $p_{t\restriction(i+2)}$ is played at move i for $i < 2 \cdot \alpha$.

Every branch in T generates a generic filter, and condition 2 ensures that each branch generates a unique filter. $\qquad\square$

2.3 WOODIN'S PROOF

We now present the more sophisticated version of Silver's proof from weaker hypotheses. Work of Gitik [4] shows that the hypotheses of Theorem 2.3.1 can be achieved from a model of $o(\kappa) = \kappa^{++}$, and it is in Part III where we show that this is optimal.

Theorem 2.3.1. *Suppose GCH holds and assume there exists an elementary embedding $j : V \to M$ with critical point κ such that ${}^\kappa M \subseteq M$ and $\kappa^{++} = (\kappa^{++})^M$. Then there is a generic extension in which κ is measurable and $2^\kappa > \kappa^+$.*

Proof. First, we argue that j may be assumed to have been derived from a (κ, κ^{++})-extender.

Claim 2.3.1. If E is a (κ, κ^{++})-extender, then the map $j_E : V \to \mathrm{Ult}(V, E)$ satisfies the hypotheses of the Theorem.

Proof. It suffices to show that if $f : \kappa \to \kappa$ is a function so that $j(f)(\kappa) = \kappa^{++}$, then $j_E(f)(\kappa) = \kappa^{++}$, and moreover that ${}^\kappa\mathrm{Ult}(V, E) \subseteq \mathrm{Ult}(V, E)$.

For the first claim, since $\kappa^{++} = (\kappa^{++})^M$, then $\kappa^{++} = [f]_{\mathrm{Ult}(V,E)}$ where $f : \alpha \mapsto \alpha^{++}$. Then $j_E(f)(\kappa) = \kappa^{++}$: that $j_E(f)(\kappa) \le \kappa^{++}$ is clear, and that $j_E(f)(\kappa) \ge \kappa^{++}$ follows from the fact that $j_E(f)(\kappa) > j(\mathrm{id})(\alpha)$ for every $\alpha < \kappa^{++}$.

For the second claim, let $\ell_E : \mathrm{Ult}(V, E) \to M$ be the factor map so that $\ell_E \circ j_E = j$, giving a commutative diagram as follows:

$$
\begin{array}{ccc}
V & \xrightarrow{\ j\ } & M \\
{\scriptstyle j_E}\big\downarrow & \nearrow {\scriptstyle \ell_E} & \\
\mathrm{Ult}(V, E) & &
\end{array}
$$

Then $\mathrm{crit}(\ell_E) > \kappa^{++}$, so that $((\kappa^{++})^\kappa)^M \leq ((\kappa^{++})^\kappa)^{\mathrm{Ult}(V,E)}$ (This is where the GCH is used.) As M is closed under functions $f : \kappa \to \kappa^{++}$ from V, so is $\mathrm{Ult}(V, E)$. Thus, $^\kappa\mathrm{Ult}(V, E) \subseteq \mathrm{Ult}(V, E)$.

This completes the proof of the claim. $\qquad\square$

In light of the previous claim, we assume that $j = j_E$ for some (κ, κ^{++})-extender E.

Now let U be the ultrafilter $U = \{X \subseteq \kappa : \kappa \in j(X)\}$, and let $i : V \to N = \mathrm{Ult}(V, U)$ be the corresponding ultrapower map. We write $j = k \circ i$ where $k : N \to M$ is given by $k([f]) = j(f)(\kappa)$, as in the following diagram:

$$
\begin{array}{ccc}
V & \xrightarrow{\ j\ } & M \\
{\scriptstyle i}\big\downarrow & \nearrow {\scriptstyle k} & \\
N & &
\end{array}
$$

Before we do any forcing, we start by analyzing the model M and the embeddings i, j, k in some detail.

Let $\lambda = (\kappa^{++})^N$. First, note that $\lambda < \kappa^{++}$ since $i(\kappa) < (2^\kappa)^+ = \kappa^{++}$. Next, notice that $\mathrm{crit}(k) = \lambda$: We just argued that $\lambda < \kappa^{++}$, and $\mathrm{crit}(k) > \kappa^+$ since $k(\lambda) = (\kappa^{++})^M = \kappa^{++} > \lambda$. Moreover, we have that $\lambda < i(\kappa)$ since $i(\kappa)$ must be inaccessible in N.

Next, we prove a lemma characterizing the model M.

Lemma 2.3.2. $M = \{k(f)(a) : a \in [\kappa^{++}]^{<\omega},\ f \in N,\ \mathrm{dom}(f) = [\lambda]^{|a|},\ \text{and } N \vDash |\mathrm{dom}(f)| \leq \kappa^{++}\}$.

Proof. By Lemma 1.4.4,

$$M = \{j(f)(a) : a \in [\kappa^{++}]^{<\omega}, \ \mathrm{dom}(f) = [\kappa]^{|a|}\}.$$

Since $j = k \circ i$, we have

$$M = \{k(g)(a) : a \in [\kappa^{++}]^{<\omega}, \ g \in N\}.$$

Since $k(\lambda) = (\kappa^{++})^M = \kappa^{++}$, we have that for all $a \in [\kappa^{++}]^{<\omega}$, $k(g)(a) = k(g \restriction \lambda^{<\omega})$.
Thus,

$$M = \{k(g)(a) : a \in [\kappa^{++}]^{<\omega}, \ g \in N, \ \mathrm{dom}(g) = [\lambda]^{|a|}\}.$$

This completes the proof of the Lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Next, we show that $j(\kappa) < \kappa^{+++}$. By Lemma 1.4.4, we have that

$$M = \{j(f)(a) : a \in [\kappa^{++}]^{<\omega}, \ \mathrm{dom}(f) = [\kappa]^{|a|}\}.$$

Since $j(f)(a) < j(\kappa)$ if and only if $a \in j(\{x : f(x) < \kappa\})$, we have that any f with
$j(f)(a) < j(\kappa)$ has value less than κ E_a-almost everywhere, we can without loss of
generality restrict attention to those $f : [\kappa]^{|a|} \to \kappa$. Thus, we can define a surjection

$$\Phi : [\kappa^{++}]^{<\omega} \times \kappa^\kappa \to j(\kappa) \ ; \ \Phi(a, f) = j(f)(a)$$

So $|j(\kappa)| \leq \kappa^{++} \cdot 2^\kappa = \kappa^{++} \cdot \kappa^+ = \kappa^{++}$. Thus $j(\kappa) < \kappa^{+++}$.

To summarize, we have the following string of inequalities:

$$\kappa^+ = (\kappa^+)^N = (\kappa^+)^M < \mathrm{crit}(k) = (\kappa^{++})^N = \lambda < i(\kappa) < \kappa^{++} = (\kappa^{++})^M < j(\kappa) < \kappa^{+++}.$$

As in Silver's proof, let $X \subseteq \kappa$ be any set of inaccessible cardinals so that $\kappa \in j(X)$,
and let $\dot{\mathbb{Q}}_\alpha = (\mathrm{Add}(\alpha, \alpha^{++}))^{V^{\mathbb{P}_\alpha}}$ if $\alpha \in X$ and $\dot{\mathbb{Q}}_\alpha = \{1\}$ otherwise. Let $\mathbb{P} = \mathbb{P}_{\kappa+1}$,
and note that $\mathbb{P}$ is isomorphic to the two-step iteration $\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa$, so that if G_κ is $\mathbb{P}_\kappa$-
generic over V and H_κ is $\mathbb{Q}_\kappa = i_{G_\kappa}(\dot{\mathbb{Q}}_\kappa)$-generic over $V[G_\kappa]$, then $V[G] = V[G_\kappa][H_\kappa]$.
Also recall, just as in Silver's proof, that $\mathbb{P}_\kappa$ is κ-c.c. of size κ, κ is inaccessible in

$V[G], V[G] \models 2^\kappa = \kappa^{++}$, and $(j(\mathbb{P}))_{\kappa+1} = \mathbb{P}$. Notice further that since G_κ is generic over V, it is also generic over M and N.

Now use the fact that $^\kappa N \subseteq N$ and $\mathbb{P}_\kappa$ is κ-c.c. to conclude, just as we did in Lemma 2.1.2, that $V[G_\kappa] \models {}^\kappa(N[G_\kappa]) \subseteq N[G_\kappa]$.

We now analyze k in detail. We would like to lift k with the generic object G_κ. But we can easily see that this is possible: since $\text{crit}(k) = (\lambda$, we have $k(\mathbb{P}_\kappa) = \mathbb{P}_\kappa$ and $k(p) = p$ for every $p \in \mathbb{P}_\kappa$. So, $k``G_\kappa \subseteq G_\kappa$, and by Lemma 2.2.1 we may extend the embedding

$$k : N \to M$$

to an embedding

$$k^+ : N[G_\kappa] \to M[G_\kappa].$$

From this point forward, we will abuse notation and write k for the original and extended embeddings.

Consider the forcing $\mathbb{Q}^{n[G_\kappa]}$, which, since $\lambda = (\kappa^{++})^{N[G_\kappa]}$, is the forcing $\text{Add}(\kappa, \lambda)^{V[G_\kappa]}$, and furthermore, let

$$h_\kappa = \{p \in \text{Add}(\kappa, \lambda)^{V[G_\kappa]} : k(p) \in H_\kappa\}.$$

Notice that if $q \in \mathbb{Q}_\kappa^{N[G_\kappa]}$, then $\text{supp}(q) \subseteq \kappa \times \mu$ for some $\mu < \lambda$, so $k(q) = q$. Hence,

$$h_\kappa = H_\kappa \restriction \lambda.$$

Now, (in $V[G_\kappa][H_\kappa]$), we wish to lift k again with the generics h_κ and H_κ. But this is easy by the previous paragraph: $k``h_\kappa = h_\kappa \subseteq H_\kappa$, and so by Lemma 2.2.1 we may lift the embedding

$$k : N[G_\kappa] \to M[G_\kappa]$$

to an embedding

$$k : N[G_\kappa][h_\kappa] \to M[G_\kappa][H_\kappa].$$

Now let $\dot{\mathbb{R}} = (\mathbb{R}_{\kappa+1, i(\kappa)})^N$, i.e. the poset so that

$$\mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa^N * \dot{\mathbb{R}} = i(\mathbb{P}_\kappa)$$

and let $\mathbb{R} = i_{G_\kappa * h_\kappa}(\dot{\mathbb{R}})$. (Note that this i denotes interpretation rather than an elementary embedding).

We now analyze the partial order $\mathbb{R}$ and the models $V[G_\kappa][h_\kappa]$ and $N[G_\kappa][h_\kappa]$. First, note that since $N[G_\kappa] \vDash$ "$\mathbb{Q}_\kappa^{N[G_\kappa]}$ is κ^+-c.c" and since h_κ is $\mathbb{Q}_\kappa^{V[G_\kappa]}$-generic over $V[G_\kappa]$ and over $N[G_\kappa]$, we may use Lemma 2.2.3 to see that $V[G_\kappa][h_\kappa] \vDash {}^\kappa N[G_\kappa][h_\kappa] \subseteq N[G_\kappa][h_\kappa]$. Furthermore, notice that $\mathbb{R}$ has size $i(\kappa)$ and that $i(\mathbb{P}_\kappa)$ is an $i(\kappa)$-c.c. poset with $i(\kappa)$-many antichains. Since $i(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \mathbb{Q}_\kappa^{N[G_\kappa]} * \mathbb{R}$, Lemma 1.2.9 says that $\mathbb{R}$ is $i(\kappa)$-c.c. We can also see that because $V[G_\kappa][h_\kappa] \vDash |X| \leq \kappa^+$ and we are using an Easton support iteration, that $N[G_\kappa][h_\kappa] \vDash$ "$\mathbb{R}$ is λ^+-closed."

By the above, we may invoke Lemma 2.2.4 to conclude there is, in $V[G_\kappa][h_\kappa]$, an $\mathbb{R}$-generic filter H over $N[G_\kappa][h_\kappa]$. Also, by the above, and the fact that every element of $M[G_\kappa][H_\kappa]$ is of the form $k(f)(a)$ for some $f \in N[G_\kappa][h_\kappa]$ such that $N[G_\kappa][h_\kappa] \vDash |\mathrm{dom}(f)| \leq \lambda$, we may invoke Lemma 2.2.2 and build, in $V[G_\kappa][H_\kappa]$, a generic filter $H^\bullet$ which is $\mathbb{R}_{\kappa+1, j(\kappa)}^M$-generic over $M[G_\kappa][H_\kappa]$.

Now we wish to lift k once more with the generic objects H and $H^\bullet$. But clearly $k"H \subseteq H^\bullet$, so we may once again extend k from an embedding

$$k : N[G_\kappa][h_\kappa] \to M[G_\kappa][H_\kappa]$$

to an embedding

$$k : N[G_\kappa][h_\kappa][H] \to M[G_\kappa][H_\kappa][H^\bullet].$$

It follows that the embeddings

$$j : V \to M$$

$$i : V \to N$$

38

can be extended (in $V[G] = V[G_\kappa][H_\kappa]$) to embeddings

$$j : V[G_\kappa] \to M[G_\kappa][H_\kappa][H^\bullet]$$

$$i : V[G_\kappa] \to N[G_\kappa][h_\kappa][H]$$

yielding a commutative diagram of now elementary embeddings

$$
\begin{array}{ccc}
V[G_\kappa] & \xrightarrow{\;\;j\;\;} & M[G_\kappa][H_\kappa][H^\bullet] \\
\downarrow{\scriptstyle i} & \nearrow{\scriptstyle k} & \\
N[G_\kappa][h_\kappa][H] & &
\end{array}
$$

Now we start the second stage of the construction, namely an extension of $V[G] = V[G_\kappa][H_\kappa]$, and in this extension, an elementary embedding extending $j : V[G_\kappa] \to M[G_\kappa][H_\kappa][H^\bullet]$. We force over $V[G]$ with the partial order $\mathbb{Q} = i(\mathbb{Q}_\kappa)$ and denote the generic object by K. Note that K is generic over $V[G]$ for the partial order $\mathbb{Q} = \mathrm{Add}(i(\kappa), i(\kappa^{++}))^{N[G_\kappa][h_\kappa][H]}$, but cannot be constructed inside $V[G]$, as $\mathbb{Q}$ has size κ^{++} but is only κ^+-closed in $V[G_\kappa][h_\kappa]$. The extension $V[G][K]$ will be the model in which we will show that $2^\kappa = \kappa^{++}$ and κ remains measurable. First, we need to show that κ^+ and κ^{++} are preserved in $V[G][K]$.

Lemma 2.3.3. $V[G] \vDash$ "$\mathbb{Q}$ is κ^+-distributive and κ^{++}-c.c."

Proof. For the first claim, we wish to use Easton's Lemma 1.1.16 with the models $V[G] = V[G_\kappa][H_\kappa]$ and $V[G_\kappa][h_\kappa]$. To do this, we need to ensure that $\mathbb{Q}$ is κ^+-closed in $V[G_\kappa][h_\kappa]$, and that the forcing which extends $V[G_\kappa][h_\kappa]$ to $V[G_\kappa][H_\kappa]$ is κ^+-c.c. The first of those two holds because $V[G_\kappa][h_\kappa] \vDash$ "$M[G_\kappa][h_\kappa][H]$ is closed under κ-sequences", and the second holds because that forcing in question is just κ-Cohen forcing. Thus, we may use Easton's Lemma 1.1.16 to conclude that $\mathbb{Q}$ is κ^+-distributive in $V[G]$.

For the second claim, let

$$\mathbb{Q}^* = \prod_{\alpha < \kappa, \text{ full support}} \mathbb{Q}_\kappa$$

Notice that $\mathbb{Q}^*$ is κ^{++}-c.c. in $V[G_\kappa]$, so it is also κ^{++}-c.c. in $V[G]$. But there is a one-to-one correspondence between antichains in $\mathbb{Q}$ and antichains in $\mathbb{Q}^*$ since conditions in $\mathbb{Q}$ are of the form $i(g)(\kappa)$, $g : \kappa \to \mathbb{Q}_\kappa$. Thus $\mathbb{Q}$ is also κ^{++}-c.c. in $V[G]$. $\qquad \square$

This Lemma shows that $V[G]$ and its extension $V[G][K]$ have the same cardinals, so that $V[G][K] \vDash 2^\kappa = \kappa^{++}$. The final step is to show that κ is measurable in $V[G][K]$, and we wish do this, just like in Silver's proof, by extending the embedding j. Note that at this point j only has domain $V[G_\kappa]$ — not even $V[G]$.

We would like to mimic the same strategy to extend j that we used to get the first extension of j, namely to transfer a generic through k. Since $\mathbb{Q}$ is sufficiently distributive as discussed in the previous Lemma, and by Lemma 1.4.4, we may use Lemma 2.2.2 to transfer K through the map k to get a filter $K^\bullet$ which is $\mathrm{Add}(j(\kappa), j(\kappa^{++}))$-generic over $M[G][H^\bullet]$. At this point we do not have enough to lift j like we did before, since we don't know that $j``H_\kappa \subseteq K^\bullet$: for $\alpha \in [\lambda, \kappa^{++})$, the αth κ-Cohen set chosen by H_κ does not belong to the model $N[G_\kappa][h_\kappa][H]$ and therefore is not an initial segment of the $j(\alpha)^{\text{th}}$ $j(\kappa)$-Cohen set chosen by $K^\bullet$, whose restriction to κ does belong to $N[G_\kappa][h_\kappa][H]$.

This is where we alter the generic object $K^\bullet$ in a clever way. We work in $V[G][K]$ and construct a suitable generic $\tilde{K}^\bullet$ by altering the elements of $K^\bullet$ to conform with j and H_κ, and we may do this by altering at most κ-many components of those elements, which will be enough by the closure of $M[G][K^\bullet]$ under κ-sequences. Let $Q = \bigcup j``H_\kappa$, so that $Q$ is a partial function from $\kappa \times j``\kappa^{++} \to 2$. Let $p \in K^\bullet$, so that $p = j(P)(a)$ for some $a \in [\kappa^{++}]^{<\omega}$ and some function $P : [\kappa]^{|a|} \to \mathbb{Q}_\kappa$ with $P \in V[G_\kappa]$. Notice that $|\mathrm{dom}(Q) \cap \mathrm{dom}(p)| \leq \kappa$: If $(\eta, j(\xi)) \in \mathrm{dom}(p)$, then by elementarity we have $(\eta, \xi) \in \mathrm{dom}(P(x))$ for some $x \in [\kappa]^{|a|}$. Now let $\tilde{p}$ be the result of altering p to

agree with Q on $\mathrm{dom}(p) \cap \mathrm{dom}(Q)$. Since $V[G] \vDash^\kappa M[G][H^\bullet] \subseteq M[G][H^\bullet]$, $\tilde{p} \in M[G]$, so $\tilde{p} \in j(\mathbb{Q}_\kappa)$. We now let

$$\tilde{K}^\bullet = \{\tilde{p} : p \in K^\bullet\}.$$

We need to show that $\tilde{K}^\bullet$ is $j(\mathbb{Q}_\kappa)$-generic over $M[G][k(H)]$. Work in $V[G_\kappa]$. Let $\lambda < \kappa$ and let D be a dense set in $\mathbb{Q}_\kappa$. Let E be the set of $p \in D$ so that the result from altering p on any set of size λ is also in D. It is easy to check that E is also dense. Return to $M[G_\kappa][H_\kappa][k(H)]$ and use the same argument with κ in place of λ to see that $\tilde{K}^\bullet$ meets every dense set in $M[G_\kappa][H_\kappa][k(H)]$.

We now have the required compatibility conditions to use Lemma 2.2.1 to lift the map

$$j : V[G_\kappa] \to M[G][H^\bullet]$$

to a map

$$j : V[G_\kappa][H_\kappa] = V[G] \to M[G][H^\bullet][\tilde{K}^\bullet].$$

This doesn't quite suffice, though, since $\tilde{K}^\bullet$ is in $V[G][K]$, not $V[G]$. But notice that we may use Lemma 1.4.4, as well as the fact that $\mathbb{Q}$ is κ^+-distributive to invoke Lemma 2.2.2 and transfer K through j to get $K^\dagger$ which is $j(\mathbb{Q})$-generic, and use Lemma 2.2.1 to lift

$$j : V[G] \to M[G][H^\bullet][\tilde{K}^\bullet]$$

again to get

$$j : V[G][K] \to M[G][H^\bullet][\tilde{K}^\bullet][K^\dagger].$$

This elementary embedding is definable in $V[G][K]$, so κ is measurable in $V[G][K]$, which completes the proof of Theorem 2.3.1.

$\square$

CHAPTER 3

THE LOWER BOUND

3.1 THE MODEL L[U]

We continue where we left off in our discussion of measurable cardinals in Section 1.3 by discussing iterated ultrapowers.

Definition 3.1.1. An *iterated ultrapower* of a well-founded inner model M by a measure $U \in M$ on a measurable cardinal κ is a sequence $\langle M_\alpha : \alpha < \beta \rangle$ defined by setting

1. $M_0 = M$.

2. $M_{\alpha+1} = \text{Ult}(M_\alpha, i_{0,\alpha}(U))$, and the ultrapower is constructed in M_α; $i_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$ is the canonical embedding, and for all $\beta < \alpha$, $i_{\beta,\alpha+1} = i_{\alpha,\alpha+1} \circ i_{\beta,\alpha}$.

3. For limit ordinals γ, M_γ is the direct limit of $\langle (M_\alpha, i_{\alpha,\beta}) : \alpha \leq \beta < \gamma \rangle$.

If $\langle M_\alpha : \alpha < \beta \rangle$ is an iterated ultrapower of M, then we call the models M_α *iterates* of M.

If M_α is well-founded then we will always identify it with its transitive collapse.

We will need the fact that every iterated ultrapower of a well-founded inner model M is well-founded.

Theorem 3.1.2. *Every iterate of an inner model of ZFC is well-founded.*

Proof. We prove the the theorem by induction on α. Suppose M is a well-founded inner model of ZFC, and suppose that M satisfies that U is a countably complete

ultrafilter. If $\alpha = 1$, then by Lemma 1.3.3, $\text{Ult}(M, U)$ is well-founded. Since all iterates of M are definable subclasses of M, we can work inside M. For any ordinal α such that M_α is well-founded, it then follows, by working inside M_α, that $M_{\alpha+1}$ is well-founded. Hence the least ordinal γ such that M_γ is ill-founded would be a limit ordinal. Fix γ to be this ordinal.

The ordinals of M_γ are not well-ordered, so let ξ be the least ordinal such that the ordinals of M_γ below $i_{0,\gamma}(\xi)$ are not well-ordered. Let $x_0, x_1, x_2, ...$ be a descending sequence of ordinals in M_γ such that $x_0 < i_{0,\gamma}(\xi)$. Since M_γ is the direct limit of $\langle M_\alpha : \alpha < \gamma \rangle$, there is an ordinal $\alpha < \gamma$ and an ordinal $\nu < i_{0,\alpha}(\xi)$ such that $x_0 = i_{\alpha,\gamma}(\nu)$. Let β be such that $\alpha + \beta = \gamma$.

In M, we know that for all $\gamma' \le \gamma$ and for all $\xi' < \xi$, the ordinals below $i_{0,\gamma'}(\xi')$ in $M_{\gamma'}$ are well-ordered. By elementarity, in M_α, we have that for all $\gamma' \le i_{0,\alpha}(\gamma)$ and for all $\xi' < i_{0,\alpha}(\xi)$, the ordinals below $i_{\alpha,\alpha+\gamma'}(\xi')$ in (a model isomorphic to) $M_{\alpha+\gamma'}$[1] are well-ordered. We may take $\gamma' = \beta$ and $\xi' = \nu$ in this statement to see that, in M_α, the ordinals below $i_{\alpha,\alpha+\beta}(\nu)$ in (a model isomorphic to) $M_{\alpha+\beta}$ are well-ordered. Since $\alpha + \beta = \gamma$ and $i_{\alpha,\gamma}(\nu) = x_0$, we have that the ordinals below x_0 in M_γ are well-ordered, which is a contradiction. $\square$

Definition 3.1.3. Let A be a set. Then the inner model $L[A]$ is defined by transfinite recursion on α as follows:

$$L_0[A] = \emptyset$$

$$L_{\alpha+1}[A] = \{X \subseteq L_\alpha[A] : X \text{ is definable over } (L_\alpha[A], \in, A \cap L_\alpha[A])\}$$

$$L_\lambda[A] = \bigcup_{\beta < \lambda} L_\beta[A] \quad \text{for } \lambda \text{ a limit ordinal.} \tag{3.3}$$

$$L[A] = \bigcup_{\alpha \in \text{ON}} L_\alpha[A]$$

---

[1]We are implicitly using the fact that $M_{\alpha+\gamma'} \cong (M_\alpha)_{\gamma'}$, where the former model is the $(\alpha + \gamma')^{\text{th}}$ iterate of M, while the latter is the γ'^{th} iterate of M_α.

Note that $L_{\alpha+1}[A]$ is defined as the collection of definable subsets of $L_\alpha[A]$ *in an expanded language* with a predicate symbol for membership in A.

If κ is a measurable cardinal and U is a measure on κ, we may form the model $L[U]$ in exactly this way.

It is not hard to show that if U is a κ-complete ultrafilter over κ, then $L[U] \vDash$ "κ is a measurable cardinal", but it may or may not be U which witnesses this: if it is not U, then one can show that $U \cap L[U]$ will be a κ-complete ultrafilter on κ in $L[U]$. If $U \in L[U]$ then actually $L[U] \vDash$ "U is a κ-complete ultrafilter on κ."

The first lemma asserts that the GCH holds at sufficiently large cardinals in $L[U]$.

Lemma 3.1.4. Suppose U is a measure on a measurable cardinal κ. Then $L[U] \vDash$ "$\forall \lambda \geq \kappa \, (2^\lambda = \lambda^+)$."

Proof. The same argument that the GCH holds in L readily adapts to $L[U]$ for $\lambda \geq \kappa$. See [5] or [6] for the details. $\square$

Remark 3.1.5. In fact, the full GCH holds in $L[U]$, but the proof is substantially less complicated for cardinals at least κ as opposed to cardinals less than κ. We isolate the former case in Lemma 3.1.4 because the argument for these cardinals is given separately in [5] and [6].

Lemma 3.1.6 (Iterability)**.** If $L_\alpha[U] \vDash$ ZFC and $\omega_1 \cup \{U\} \subseteq L_\alpha[U]$, then every iterated ultrapower of $L_\alpha[U]$ is well-founded.

Proof. The proof follows similarly to Lemma 3.1.2, but a bit more care must be taken because $L_\alpha[U]$ is set-sized. See [6] for the details. $\square$

Remark 3.1.7. From this point forward, we will write $\kappa^{(\alpha)}$ for the ordinal $i_{0,\alpha}(\kappa)$. Note that this ordinal is also $\mathrm{crit}(i_{0,\alpha})$.

Lemma 3.1.8.   1. $\gamma < \kappa^{(\alpha)}$, then $i_{\alpha,\beta}(\gamma) = \gamma$ for all $\beta \geq \alpha$.

   2. The sequence $\langle \kappa^{(\alpha)} : \alpha \in \mathrm{Ord} \rangle$ is increasing and continuous.

Proof. For the first part, it suffices to give the proof for $\alpha = 0$. We proceed by induction on β. If $\beta = 1$, then $i_{0,1}(\gamma) = \gamma$ for all $\gamma < \kappa$ since $\kappa = \mathrm{crit}(i_{0,1})$. For the successor step, we may assume $i_{0,\beta}(\gamma) = \gamma$. Then clearly $i_{0,\beta+1}(\gamma) = (i_{0,\beta} \circ i_{\beta,\beta+1})(\gamma) = \gamma$ because $\gamma < \kappa^{(\beta)} = \mathrm{crit}(i_{0,\beta+1})$. The limit stage follows easily.

For the second statement, the sequence is clearly increasing: for each α, $\kappa^{(\alpha+1)} = i_{\alpha,\alpha+1}(\kappa^{(\alpha)}) > \kappa^{(\alpha)}$. To show that the sequence is continuous, let γ be a limit ordinal. We wish to show that $\kappa^{(\gamma)} = \lim_{\alpha \to \gamma} \kappa^{(\alpha)}$. If $\gamma' < \kappa^{(\gamma)}$, then $\gamma' = i_{\alpha,\gamma}(\delta)$ for some $\alpha < \gamma$ and $\delta < \kappa^{(\alpha)}$. Hence $\gamma' = \delta$ and so $\gamma < \kappa^{(\alpha)}$. □

Lemma 3.1.9. Suppose that U is a measure on κ and $U \in L[U]$.] Let A be a set (in V, not necessarily in $L[U]$) of ordinals of size at least κ^+ and let θ be a large enough cardinal so that $U \in L_\theta[U]$ and $A \subseteq L_\theta[U]$. Let $M = \mathcal{H}^1(\kappa \cup A)$ be the Σ_1-Skolem hull of $\kappa \cup A$ as computed in $\langle L_\theta[U], \in, U \rangle$. Then $\mathcal{P}(\kappa) \cap L[U] \subseteq M$, so that for any $X \in \mathcal{P}(\kappa) \cap L[U]$, we have that

$$X = t^M(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n)$$

for some Skolem term $t, \xi_1, ..., \xi_m \in \kappa$, and $\eta_1, ..., \eta_n \in A$.

Proof. Let

$$\pi : M \to N$$

be the Mostowski collapsing isomorphism. Then clearly $\pi(X) = X$ for every $X \in \mathcal{P}(\kappa) \cap M$. Hence, $N = \langle N, \in, U \cap N \rangle$, so $N = L_\eta[U]$ for some η. We have that $\eta \geq \kappa^+$ since $|A| \geq \kappa^+$. By the same proof as Lemma 3.1.4, we have that $(P(\kappa))^{L[U]} \subseteq L_{\kappa^+}[U]$, so $\mathcal{P}(\kappa) \cap L[U] \subseteq N$. Hence, $\mathcal{P}(\kappa) \cap L[U] \subseteq M$. □

Lemma 3.1.10. Suppose κ is a measurable cardinal and U is a measure on κ. Then

1. If λ is a cardinal greater than 2^κ, then $i_{0,\lambda}(\kappa) = \lambda$.

2. If α is an ordinal and λ is a strong limit cardinal greater than α, and if $\mathrm{cf}(\lambda) > \kappa$, then $i_{0,\alpha}(\lambda) = \lambda$.

45

Proof. For the proof of both parts, we will use the fact that for every ordinal ξ, η, the ordinals below $i_{0,\xi}(\eta)$ are represented by functions with finite support from κ^ξ into η, hence $|i_{0,\xi}(\eta)| \le |\xi| \cdot |\eta|^\kappa$.

For the first statement, we have that $i_{0,\alpha}(\kappa) = \lim_{\xi \to \alpha} i_{0,\xi}(\kappa)$, and for each $\xi < \alpha$, $|i_{0,\xi}(\kappa)| \le |\xi| \cdot 2^\kappa < \alpha$. Hence, $i_{0,\kappa}(\kappa) = \alpha$.

For the second statement, since $\mathrm{cf}(\lambda) > \kappa$, every function $f : \kappa^\alpha \to \lambda$ with finite support is bounded below λ. Let $\gamma < \lambda$ be a bound, so that $f(\vec{x}) < \gamma$ for all $\vec{x} \in \kappa^\alpha$. Hence, $i_{0,\alpha}(\lambda) = \lim_{\gamma < \lambda} i_{0,\alpha}(\gamma)$. Since λ is strong limit, we have $|i_{0,\alpha}(\gamma) < \lambda$ for all $\gamma < \lambda$ and hence $i_{0,\alpha}(\lambda) = \lambda$. $\square$

Lemma 3.1.11. Let U be a measure on a measurable cardinal κ and suppose $U \in L[U]$. Let $\lambda > \kappa^+$ be a regular cardinal and let $\mathcal{C}_\lambda$ be the club filter on λ. Then

1. $i_{0,\lambda}(U) = \mathcal{C}_\lambda \cap (L[U])_\lambda$, where $(L[U])_\lambda$ denotes the λth iterated ultrapower of $L[U]$ by the measure U.

2. $(L[U])_\lambda = L[\mathcal{C}_\lambda]$.

Proof. First, note that by Lemmas 3.1.4 and 3.1.10, we have that $i_{0,\lambda}(\kappa) = \lambda$. Let $U^{(\lambda)} = i_{0,\lambda}(U)$ and let $M = (L[U])_\lambda$. If $X \in U^{(\lambda)}$, then X contains a club, and so $X \in \mathcal{C}_\lambda$. It follows that $U^{(\lambda)} = \mathcal{C}_\lambda \cap (L[U])_\lambda$.

The second claim follows from the first part, since

$$(L[U])_\lambda = L[U^{(\lambda)}] = L[\mathcal{C}_\lambda \cap (L[U])_\lambda] = L[\mathcal{C}_\lambda].$$

$\square$

The next two theorems describe the canonicity of $L[U]$: κ is the only measurable cardinal in $L[U]$, and if W is a measure on κ and $W \in L[W]$ then actually $L[U] = L[W]$.

Theorem 3.1.12. *If U is a measure on a measurable cardinal κ and $U \in L[U]$, then κ is the only measurable cardinal in $L[U]$.*

Proof. Suppose for the sake of contradiction that

$$L[U] \vDash \text{``}\lambda \neq \kappa \text{ and } \kappa \text{ and } \lambda \text{ are both measurable cardinals.''}$$

Let W be a measure on λ and let $j : L[U] \to \text{Ult}(L[U], W)$ be the induced ultrapower embedding in $L[U]$. We shall prove that $\text{Ult}(L[U], W) = L[U]$, getting a contradiction since $U \notin \text{Ult}(L[U], W)$.

Notice that $\text{Ult}(L[U], W) = L[j(U)]$. If $\lambda > \kappa$, then $j(U) = U$ and so $\text{Ult}(L[U], W) = L[U]$, a contradiction. Hence, suppose $\lambda < \kappa$.

Let X be the set of inaccessible cardinals strictly between κ and λ. Since $X \in U$, we have $j(\kappa) = \kappa$ and $j(\alpha) = \alpha$ for every $\alpha \in X$. We shall show that $j(U) = U \cap \text{Ult}(L[U], W)$. That $j(U) \supseteq U \cap \text{Ult}(L[U], W)$ is clear. For the reverse inclusion, suppose $A \in j(U)$ is represented by a function $f : \lambda \to U$. Let $Y = \bigcap_{\xi < \lambda} f(\xi)$; we have $Y \in U$, and so clearly $j(Y) \subseteq A$ since $A \in j(U)$. Now if α is a cardinal in $Y \cap X$, then we showed that $j(\alpha) = \alpha$, and so $A \supseteq j(Y) \supseteq j\text{``}(Y \cap X) = Y \cap X \in U$, hence $A \in U$.

Thus $j(U) = U \cap \text{Ult}(L[U], W)$, and we have

$$\text{Ult}(L[U], W) = L[j(U)] = L[U \cap \text{Ult}(L[U], W)] = L[U],$$

a contradiction. Hence $\lambda = \kappa$. □

Theorem 3.1.13. *Suppose U and W are measures on a measurable cardinal κ and suppose $U \in L[U]$ and $W \in L[W]$. Then $U = W$, hence $L[U] = L[W]$.*

Proof. We show $U \subseteq W$, which implies that $L[U] \subseteq L[W]$. The other direction follows by symmetry. To this end, suppose that $X \subseteq \kappa$ and $X \in U$. We need to show that $X \in W$.

Let $\lambda \geq \kappa^+$ be a regular cardinal and let $\mathcal{C}_\lambda$ be the club filter on λ. Let

$$i : L[U] \to M = (L[U])_\lambda$$

47

and

$$j : L[W] \to N = (L[W])_\lambda$$

be the iterated ultrapowers from $L[U]$ and $L[W]$ to their λ^{th} iterates.

By Lemma 3.1.11, $M = N = L[\mathcal{C}_\lambda]$, and $i(U) = j(U) = \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]$.

By Lemma 3.1.10, there exists a set of ordinals A of size at least κ^+ such that all $\gamma \geq \lambda$ for all $\gamma \in A$, and that $i(\gamma) = j(\gamma)$ for all $\gamma \in A$. Let θ be a cardinal greater than all $\gamma \in A$ so that $i(\theta) = j(\theta) = \theta$.

By Lemma 3.1.9, $X \in \mathcal{H}^1(\kappa \cup A)$ as computed in $\langle L_\theta[U], \in, U \rangle$, and there exists a Skolem term t, ordinals $\xi_1, ..., \xi_m < \kappa$ and ordinals $\eta_1, ..., \eta_n \in A$ such that $X = t^{\langle L_\theta[U], \in, U \rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, U)$. Similarly, let $Y \in L_\theta[W]$ be such that $Y = t^{\langle L_\theta[W], \in, W \rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, W)$. We will show that $Y \in W$ and $X = Y$, completing the proof.

Claim 3.1.1. $i(X) = j(Y)$.

Proof. We know that $X = t^{\langle L_\theta[U], \in, U \rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, U)$. By applying i, we see that

$$i(X) = t^{\langle L_\theta[\mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]], \in, \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda] \rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]).$$

We also know that $Y = t^{\langle L_\theta[W], \in, W \rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, W)$. By applying j, we see that

$$j(Y) = t^{\langle L_\theta[\mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]], \in, \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda] \rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, \mathcal{C}_\lambda \cap L[\mathcal{C}_\lambda]).$$

Note that we used the fact that j does not move any parameters in the Skolem term t. Hence, $i(X) = j(Y)$. This completes the proof of the claim. $\square$

By Lemma 3.1.8, we have $X = i(X) \cap \kappa$ and $Y = j(Y) \cap \kappa$. Hence, $X = Y$. This completes the proof of the Theorem. $\square$

Corollary 3.1.14. If κ is a measurable cardinal, U is a measure on κ, and $U \in L[U]$ then U is the only normal ultrafilter over κ in $L[U]$.

Proof. Suppose W is a normal ultrafilter over κ in $L[U]$. Let $\bar{W} = W \cap L[W]$. Then $\bar{W} \in L[\bar{W}]$, which means that $\bar{W} = U$, so actually $W = U$. $\qquad\qquad$ $\square$

Thus, there is a one-to-one correspondence between measurable cardinals and models of the form $L[U]$. In this light, we call $L[U]$ *the κ-model* if $U \in L[U]$ and U is a measure on κ. The next theorem shows much more, namely that every κ-model is an iterated ultrapower of the κ-model corresponding to the least measurable cardinal.

Theorem 3.1.15. *Suppose $L[U]$ is the κ-model and $L[W]$ is the λ-model, with $\kappa < \lambda$. Then there exists an ordinal α so that $L[W] = \mathrm{Ult}^{(\alpha)}(L[U], U)$.*

Proof. Let α be the unique ordinal so that if $i_{0,\alpha} : L[U] \to (L[U])_\alpha$ is the αth iterated ultrapower of $L[U]$ by the measure U, then

$$\kappa^{(\alpha)} \leq \lambda < \kappa^{(\alpha+1)}.$$

It suffices to show that $\lambda = \kappa^{(\alpha)}$, whence the Theorem follows from the uniqueness of $i_{0,\alpha}(U)$. We know that $\lambda < \kappa^{(\alpha+1)}$, so for the sake of contradiction suppose $\kappa^{(\alpha)} < \lambda < \kappa^{(\alpha+1)}$.

Let $j : L[W] \to \mathrm{Ult}(L[W], W)$ be the canonical ultrapower embedding, let $\mu = |\lambda|^{++}$, and let $\mathcal{C}_\mu$ be the club filter on μ. Since $L[W] \vDash \mathrm{GCH}$, we have $j(\mu) = \mu$. In $L[W]$, we have that $\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]$ is the μth iterate of W, and in $L[j(W)]$, $\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]$ is the $j(\mu)^{\mathrm{th}}$ iterate of $j(W)$. Hence, $j(\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]) = \mathcal{C}_\mu \cap L[\mathcal{C}_\mu]$.

Let $f : \kappa^{(\alpha)} \to \kappa^{(\alpha)}$ be a function in $L[i_{0,\alpha}(U)]$ which represents λ in $\mathrm{Ult}(L[i_{0,\alpha}(U)], i_{0,\alpha}(U))$. Since $i_{0,\alpha}(U)$ is normal, the diagonal function represents $\kappa^{(\alpha)}$, hence $(i_{0,\alpha+1}(f))(\kappa^{(\alpha)}) = \lambda$. Let $i_{0,\mu} : L[i_{0,\alpha}(U)] \to (L[i_{0,\alpha}(U)])_\mu = L[\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]]$. It is clear that $(i_{0,\mu}(f))(\kappa^{(\alpha)}) = \lambda$.

Now let A be a set of ordinals so that $|A| = \kappa^+$, $\xi > \mu$ for every $\xi \in A$, and $i_{0,\mu}(\xi) = j(\xi) = \xi$ for every $\xi \in A$. Let θ be a cardinal greater than all $\xi \in A$ such that $i_{0,\mu}(\theta) = j(\theta) = \theta$.

Note that λ is definable in $L_\theta[\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]]$ from $A \cup \kappa^{(\alpha)} \cup \{\kappa^{(\alpha)}\} \cup \{\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]\}$, hence we may use Lemma 3.1.9 to find a Skolem term t, ordinals $\xi_1, ..., \xi_m < \kappa^{(\alpha)}$, and ordinals $\eta_1, ..., \eta_n \in A$ such that

$$\lambda = t^{\langle L_\theta[\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]], \in, \mathcal{C}_\mu \cap L[\mathcal{C}_\mu]\rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, \mathcal{C}_\mu \cap L[\mathcal{C}_\mu], \kappa^{(\alpha)}).$$

Apply the embedding j to the above to see that

$$j(\lambda) = t^{\langle L_\theta[\mathcal{C}_\mu \cap L[\mathcal{C}_\mu]], \in, \mathcal{C}_\mu \cap L[\mathcal{C}_\mu]\rangle}(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n, \mathcal{C}_\mu \cap L[\mathcal{C}_\mu], \kappa^{(\alpha)}),$$

where we use the fact that j does not move any parameters in the Skolem term. Hence, $j(\lambda) = \lambda$, which is a contradiction. Thus, $\lambda = \kappa^{(\alpha)}$, which completes the proof of the Theorem. $\square$

Corollary 3.1.16. If $L[U]$ is the κ-model and $L[W]$ is the λ-model with $\lambda > \kappa$, then $L[W] \subseteq L[U]$ and $L[W]$ is definable in $L[U]$ from λ.

3.2   THE MITCHELL ORDER AND MITCHELL RANK

There are many natural ways of strengthening measurability to climb up the large cardinal hierarchy. One way, as explored in the preliminary chapter, is to demand some amount of closure of the target model of the elementary embedding. For example, we saw before that a cardinal κ is λ-supercompact if $j(\kappa) > \lambda$ and $^\lambda M \subseteq M$.

Another approach is to focus on the measurability of κ in the target model. If κ is measurable, U is a κ-complete ultrafilter on κ, and $j : V \to M$ is the induced embedding, then κ may or may not be measurable in M, since U is never a member of M. If κ were to remain measurable in M, there would have to be, in V, some other κ-complete ultrafilter W on κ which is not destroyed by the embedding j. It is not hard to show that if κ is a measurable cardinal which remains measurable in M, then there are κ measurable cardinals below κ, so this really is a stronger condition to impose.

Motivated by this, we may define the Mitchell ordering on κ-complete ultrafilters on a measurable cardinal κ.

Definition 3.2.1. If κ is a measurable cardinal and U and W are κ-complete ultrafilters on κ, then we say $U \triangleleft W$ if $U \in \mathrm{Ult}(V, W)$.

We may now analogously define the Mitchell order of a measurable cardinal κ.

Definition 3.2.2. If κ is a measurable cardinal and X is the collection of measures on κ, then $o(\kappa)$ is the height of $(X, \triangleleft)$

We now bookmark some key properties of the Mitchell order.

Lemma 3.2.3. 1. $\triangleleft$ is an irreflexive, transitive, well-founded relation.

  2. For every measurable cardinal $\kappa, o(\kappa) \leq (2^{\kappa})^{+}$.

Proof. See [5] or [6]. □

Assuming GCH, clause 2 of Lemma 3.2.3 says that $o(\kappa) \leq \kappa^{++}$ for every measurable cardinal κ. The existence of a cardinal κ of this maximal Mitchell rank is the exact strength of the GCH failing at a measurable cardinal. Indeed, for the remainder of this thesis we aim to prove the following theorem:

Theorem 3.2.4. *If there is a measurable cardinal κ with $2^{\kappa} > \kappa^{+}$, then there is an inner model with a measurable cardinal λ such that $o(\lambda) = \lambda^{++}$.*

The model K in the proof of Theorem 3.2.4 ends up being an analogue of $L[U]$ which can include cardinals of high Mitchell rank.

3.3 FINE STRUCTURE

In this section, we introduce the basic definitions behind fine structure theory. While comfort with fine structure theory would be helpful in making some of the forthcoming

ideas clearer, the reader who is not familiar with fine structure theory already should feel free to skip this section, as we do not assume any familiarity in forthcoming sections. Nonetheless, we review the definitions here for consistency's sake. For a more complete introduction to fine structure theory, we refer the reader to [16].

Definition 3.3.1. Let A be a (possibly not proper) class. A function $f : V^k \to V$ for some $k < \omega$, is called *rudimentary in A* (or rud_A) if it is generated by the following schemata:

$$f\left(\langle x_1, \ldots, x_k \rangle\right) = x_i$$

$$f\left(\langle x_1, \ldots, x_k \rangle\right) = x_i \backslash x_j$$

$$f\left(\langle x_1, \ldots, x_k \rangle\right) = \{x_i, x_j\}$$

$$f\left(\langle x_1, \ldots, x_k \rangle\right) = h\left(g_1\left(\langle x_1, \ldots, x_k \rangle\right), \ldots, g_\ell\left(\langle x_1, \ldots, x_k \rangle\right)\right)$$

$$f\left(\langle x_1, \ldots, x_k \rangle\right) = \bigcup_{y \in x_1} g\left(\langle y, x_2, \ldots, x_k \rangle\right)$$

$$f(x) = x \cap A$$

f is called *rudimentary* (or rud) if f is $\mathrm{rud}_\emptyset$.

Definition 3.3.2. If X is a set and A is a (possibly not proper) class, then we denote by $\mathrm{rud}_A(X)$ the rud_A closure of X, i.e. the set

$$X \cup \{f\left(\langle x_1, \ldots, x_k \rangle\right) \mid f \text{ is } \mathrm{rud}_A \text{ and } x_1, \ldots, x_k \in X\}$$

Definition 3.3.3. Let A be a (possibly not proper) class. A relation $R \subseteq V^k$ for some $k < \omega$ is called rudimentary in A (or rud_A) if there is a rud_A function $f : V^k \to V$ such that $R = \{\vec{x} \mid f(\vec{x}) \neq \emptyset\}$. R is called rudimentary (or, rud) if R is $\mathrm{rud}_\emptyset$.

We may now define the $J_\alpha[A]$ hierarchy, indexed by limit ordinals.

Definition 3.3.4. Let A be a set or a proper class.

$$J_0[A] = \emptyset$$

$$J_{\alpha+\omega}[A] = \mathrm{rud}_A\left(J_\alpha[A] \cup \{J_\alpha[A]\}\right)$$

$$J_{\omega\lambda}[A] = \bigcup_{\alpha<\lambda} J_{\omega\alpha}[A] \ \text{ for } \lambda \text{ a limit ordinal}$$

$$L[A] = \bigcup_{\alpha\in\mathrm{On}} J_{\omega\alpha}[A]$$

We then define J_α to be the structure $J_\alpha[\emptyset]$

In all of the following definitions until the end of this section, assume that α is a limit ordinal, and that $M = (J_\alpha, A)$ is *amenable*, that is $A \cap x \in J_\alpha$ for all $x \in J_\alpha$.

Definition 3.3.5. The Σ_1-*projectum* ϱ_1^M of an amenable structure $M = (J_\alpha, A)$ is the least ordinal ϱ such that there is a Σ_1 subset x of ϱ which is not a member of J_α, but is Σ_1-definable in M using a finite set $p \subseteq \alpha$ as a parameter.

Definition 3.3.6. The Σ_1-*standard parameter* p_1^M of M is the least finite sequence $p \in [\alpha]^{<\omega}$ of ordinals such that there is some set $x \subseteq \varrho_1^M$ so that $x \notin J_\alpha$, but x is Σ_1-definable in M from parameters in $\varrho_1^M \cup p$.

The ordering of the parameters is lexicographical on descending sequences of ordinals; that is, $p < p'$ if $\max(p \triangle p') \in p'$.

Definition 3.3.7. The Σ_1-*master code* is the set A_1^M of pairs $(\ulcorner\varphi\urcorner, \xi)$ such that $\xi < \varrho_1^M$ and $\ulcorner\varphi\urcorner$ is the Gödel number of a Σ_1 formula φ over M, with parameter p_1^M, such that $M \models \varphi(\xi)$.

Definition 3.3.8. The Σ_1-*Skolem function* h_1^M of M is defined as follows: fix an enumeration $\langle \exists x \varphi_n(x) : n < \omega \rangle$ of the Σ_1 formulas of set theory. Then $h_1^M(\langle n, x \rangle)$ is defined if and only if there are z and y such that $M \models \varphi_n\left(x, y, z, p_1^M\right)$. In this case $h_1^M(\langle n, x \rangle) = y$ where (α', z, y) is the lexicographically least triple such that $(J_{\alpha'}, A \cap \alpha') \models \varphi_n\left(x, y, z, p_1^M\right)$.

It should be noted that, unlike L_α, J_α is closed under finite sequences, even when α is a successor ordinal. Thus, a finite set of ordinals $\{\alpha_0, ..., \alpha_k\}$ can be treated as a single parameter $\langle \alpha_0, ..., \alpha_k \rangle$.

Definition 3.3.9. The Σ_1-*code* $\mathfrak{C}_1(M)$ of M is the structure $(J_{\varrho_1^M}, A_1^M)$.

We now turn to consider the fine structure for $n > 1$. The central theme of fine structure is that it is never necessary to deal directly with Σ_{n+1} definability for any n greater than zero; instead a Σ_{n+1} formula is reduced to an equivalent Σ_1 formula over the Σ_n-code of J_α. The definition of the Σ_n-code $\mathfrak{C}_n(J_\alpha)$ is itself a good example of this theme.

Definition 3.3.10. We define the Σ_n-*codes* of J_α by recursion on $n < \omega$. We set $\mathfrak{C}_0(J_\alpha) = (J_\alpha, \emptyset)$, and for $n \geq 0$,

$$\varrho_{n+1}^{J_\alpha} = \varrho_1^{\mathfrak{C}_n(J_\alpha)} \qquad p_{n+1}^{J_\alpha} = p_1^{\mathfrak{C}_n(J_\alpha)} \qquad h_{n+1}^{J_\alpha} = h_1^{\mathfrak{C}_n(J_\alpha)}$$
$$A_{n+1}^{J_\alpha} = A_1^{\mathfrak{C}_n(J_\alpha)} \qquad \mathfrak{C}_{n+1}(J_\alpha) = \mathfrak{C}_1(\mathfrak{C}_n(J_\alpha))$$

Finally, the *projectum* of J_α is defined to be $\text{proj}(J_\alpha) = \varrho^{J_\alpha} = \inf_n \varrho_n^{J_\alpha}$. Since the sequence of projecta $\langle \varrho_n^{J_\alpha} : n < \omega \rangle$ is nonincreasing, $\varrho_n^{J_\alpha} = \text{proj}(J_\alpha)$ for all sufficiently large $n < \omega$.

Definition 3.3.11. The structure $M = (J_\alpha, A)$ is *1-sound* if $J_\alpha = h_1^M \text{``} \varrho_1^M$, and M is *n-sound* if it is $(n-1)$-sound and $\mathfrak{C}_{n-1}(M)$ is 1-sound. M is *sound* if it is n-sound for every $n < \omega$.

3.4 DEFINING THE CORE MODEL K

This section takes on the Herculean task of introducing Mitchell's core model K below $o(\kappa) = \kappa^{++}$, and is a departure in exposition compared to the rest of the thesis. We aim to fill a gap in the literature by giving an introduction to inner model theory for the non-inner-model-theorist. There are two ways in which this gap manifests

itself. For one, very few resources exist which only give a cursory overview of the core model for the non-specialist. Often times, an upper bound of an equiconsistency result is established by forcing over the core model for that particular large cardinal property, and it is usually the case that the inner model theoretic particulars are not important: only the most important properties which K enjoys. Secondly, there is a gap in the literature specifically at the level of $o(\kappa) = \kappa^{++}$. Quite a few introductory texts have been written both higher and lower down on the large cardinal hierarchy, up to $o(\kappa) = 2$ [2] and up to one Woodin cardinal[3]. Very little material exists at the level of $o(\kappa) = \kappa^{++}$. This is the role of this section. We aim to hit a delicate middle-ground in sophistication: we do not want so many details missed that there is little to be gained, yet we do not want to replicate the notoriously high barrier of entry of the current literature. The author recommends [14] for an introduction to contemporary inner model theory at the level of Woodin cardinals, and wishes them luck.

More specifically, there are two completely different approaches to building mice and the core model which serve very different purposes. The so-called "old-school" inner model theory is that of its origins, and is how Dodd, Jensen, and Mitchell constructed the first core models, including the one discussed here. It is the author's opinion that this approach is much more conceptually intuitive and easier to motivate. However, it became apparent during the development of inner model theory that this approach does not generalize to higher core models, so the approach was abandoned in favor of the modern approach. The modern approach to mice has the advantage of easily generalizing to all of the core models which have been constructed to this day, yet it suffers from the facts that it is both much more conceptually challenging, and also much more difficult to motivate when we are further down the large cardinal hierarchy, at $o(\kappa) = \kappa^{++}$. That all said, we adopt the *old* approach as in Mitchell's

---

[2]see [13] and chapters 1-4 of [16]
[3]see [14], [15], or chapters 4-7 of [16]

[9], [10], and [11] for expository purposes, but the reader should note that mice and the core model are constructed very differently when Woodin cardinals need to be accomodated. Much of the remainder of the thesis follows these papers, with typos corrected, fine-structural details omitted, and plenty of additional examples and motivation.

We now explain the organization of the next two sections, and give the reader a guide to navigating them. Sections 3.5 and 3.6 are to be read concurrently. Section 3.5 only covers the definitions involved in constructing K and statements of the basic theorems which show that the construction can be done, while Section 3.6 is devoted to giving as many examples as possible to demystify and bring together the definitions from Section 3.5. In other words, the reader is encouraged to turn to Section 3.6 as necessary when they wish to see various examples of mice and the core model being constructed.

We start by introducing a piece of terminology which will make referring to the kinds of sequences used to build the core model much more convenient.

Definition 3.4.1. A *double-index sequence* is a function $\mathcal{U}$ whose domain is of the form

$$\{(\kappa, \beta) : \kappa < \ell(\mathcal{U}) \text{ and } \beta < o^{\mathcal{U}}(\kappa)\},$$

where $\ell(\mathcal{U})$ is a cardinal and $o^{\mathcal{U}}$ is a function mapping cardinals $\kappa < \ell(\mathcal{U})$ to ordinals.

If $\mathcal{U}$ is a double-index sequence and $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U})$, we write $\mathcal{U} \upharpoonright (\kappa, \beta)$ for the restriction of $\mathcal{U}$ to the set

$$\{(\kappa', \beta') \in \mathrm{dom}(\mathcal{U}) : \kappa' < \kappa \text{ or } (\kappa' = \kappa \text{ and } \beta' < \beta)\},$$

and we write $\mathcal{U} \upharpoonright \geq (\kappa, \beta)$ for the restriction of $\mathcal{U}$ to the set

$$\{(\kappa', \beta) \in \mathrm{dom}(\mathcal{U}) : \kappa' > \kappa \text{ or } (\kappa' = \kappa \text{ and } \beta' \geq \beta)\},$$

so that for each $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U})$,

$$\mathcal{U} = (\mathcal{U} \restriction (\kappa, \beta))^\frown (\mathcal{U} \restriction \geq (\kappa, \beta)).$$

Furthermore, if ξ is any ordinal, we write $\mathcal{U} \restriction \xi$ and $\mathcal{U} \restriction \geq \xi$ for the restrictions $\mathcal{U} \restriction (\kappa, \beta)$ and $\mathcal{U} \restriction \geq (\kappa, \beta)$ respectively, where (κ, β) is the least pair ordered lexicographically with $\kappa \geq \xi$ and $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U})$.

We are now ready to introduce the key property we require double-index sequences of ultrafilters to have.

Definition 3.4.2. A *coherent sequence of measures* is a double-index sequence $\mathcal{U}$ such that

1. If $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U})$ then $\mathcal{U}(\kappa, \beta)$ is a normal ultrafilter on κ.

2. (Coherence) If $(\kappa, \beta) \in \mathrm{dom}(\mathcal{U})$, then $(i^{\mathcal{U}(\kappa,\beta)}(\mathcal{U})) \restriction \kappa + 1 = \mathcal{U} \restriction (\kappa, \beta)$.

From this point forward, we will use a generalization of the iterated ultrapowers from Section 3.1. Instead of taking an iterated ultrapower by successive images of the same ultrafilter, we now define what it means to take an iterated ultrapower of a model M by a coherent sequences of measures $\mathcal{U}$.

Definition 3.4.3. An *iterated ultrapower* of a model M is any member of a sequence $\langle M_\alpha : \alpha < \beta \rangle$ constructed inductively on α as follows:

$M_0 = M$,

$M_{\alpha+1} = \mathrm{Ult}(M_\alpha, U_\alpha)$ where $U_\alpha \in M_\alpha$ is a $\kappa^{(\alpha)}$-complete nonprincipal ultrafilter on $\kappa^{(\alpha)}$ for some cardinal $\kappa^{(\alpha)} \in M_\alpha$,

$M_\gamma = \varinjlim_{\alpha < \gamma} \langle (M_\alpha, i_{\alpha,\beta}) : \alpha < \beta < \gamma \rangle$, where the maps $i_{\alpha,\beta}$ are defined in the usual way.

An iterated ultrapower of a model M by a coherent sequence of measures $\mathcal{U}$ is a sequence as in Definition 3.4.3, except $M_{\alpha+1}$ is formed by an ultrafilter which is the image of a member of $\mathcal{U}$.

57

From this point forward, we will use the terms "sequence of measures," "ultrafilter sequence," and "sequence of ultrafilters" interchangeably, and we will also assume all sequences of measures are coherent unless otherwise specified.

Definition 3.4.4. If $\mathcal{U}$ is a double-index sequence, then a $\mathcal{U}$-*mouse* is a model $M = J_\alpha[\mathcal{V}]$ for some double-index sequence $\mathcal{V}$ such that for some ordinal ξ with $\ell(\mathcal{U}) \leq \xi \leq \alpha$ and some parameter $p \in [\alpha - \xi]^{<\omega}$,

1. $J_\alpha[\mathcal{V}] = \mathcal{H}_1^{J_\alpha[\mathcal{V}]}(\xi \cup p)$, the Σ_1-Skolem hull of $\xi \cup p$ as computed in $J_\alpha[\mathcal{V}]$,

2. $\mathcal{V} \upharpoonright \xi = \mathcal{U}$, and

3. $\mathcal{V} \upharpoonright \geq \xi$ is an ultrafilter sequence in $J_\alpha[\mathcal{V}]$ such that every iterated ultrapower is well-founded.

If M is a $\mathcal{U}$-mouse then ξ_M is the least value of ξ for which the first clause is satisfied and p_M is the least value of p (using the lexicographical ordering on descending enumerations) such that the first clause is satisfied with $\xi = \xi_M$.

Remark 3.4.5. We make two important remarks about the above definition.

1. The reader should be careful not to confuse p_M with the Σ_1-standard parameter p_1^M (see Definition 3.3.6). While ξ_M is very similar to the Σ_1-projectum ϱ_1^M and p_M is very similar to the Σ_1-standard parameter p_1^M, it is not necessarily true that $\xi_M = \varrho_1^M$ or that $p_M = p_M^1$ (See Example 3.5.5). Despite this, the reader is encouraged to cautiously interpret ξ_M as kind of like the Σ_1-projectum of M.

2. Notice that the third condition puts no restriction on $\mathcal{U}$, other than it being a double-index sequence, so it need not be an ultrafilter sequence. Later we will use mice to define recursively a sequence $\mathcal{U}$ so that $\mathcal{U}$ is an ultrafilter sequence in $L[\mathcal{U}]$.

3. We will not be proving clause 3 for any examples of mice we give. We proved in Lemma 3.1.2 that certain iterated ultrapowers of certain levels of $L[U]$ are well-founded, but for any other mice, we leave the question of iterability to the references.

One key fact about the $\mathcal{U}$-mice is that they can be compared, much like initial segments of $L[U]$ can be compared as in Lemma 3.1.15

Definition 3.4.6. Let $M = J_\alpha[\mathcal{V}]$ and $N = J_\beta[\mathcal{W}]$ be $\mathcal{U}$-mice. Then $M < N$ if there are iterated ultrapowers $i : M \to J_{\alpha'}[\mathcal{V}']$ and $j : N \to J_{\beta'}[\mathcal{W}']$ with $i \restriction \xi_M$ and $j \restriction \xi_N$ equal to the identity such that $\alpha' \le \beta', \mathcal{W}' \restriction \alpha' = \mathcal{V}'$, and $(\alpha', \xi_M, i(p_M)) < (\beta', \xi_N, j(p_N))$ lexicographically.

We have an analogue of Lemma 3.1.15 in this context where we have coherent sequences of measures instead of only one measure.

Lemma 3.4.7 (Comparison). $<$ is a well-ordering of the $\mathcal{U}$-mice.

Now, we may introduce the construction of the core model. Notice that the following definition gives, for each double-index sequence $\mathcal{U}$, a corresponding core model $K(\mathcal{U})$. It may seem confusing to have referred to "the core model" in this light, but later we will define a particular sequence of measures $\mathcal{U}_{MAX}$ which contains all of the "large cardinal information in V below $o(\kappa) = \kappa^{++}$," and the model $K(\mathcal{U}_{MAX})$ will be "the" core model. This sequence $\mathcal{U}_{MAX}$ is relative to V, and has different measures in it depending on how many measurable cardinals are in V. While the sequence $\mathcal{U}_{MAX}$ contains all measures on measurable cardinals of Mitchell order at most κ^{++}, $K(\mathcal{U}_{MAX})$ does not necessarily equal V, since V may contain large cardinals of much higher consistency strength, such as Woodin cardinals or supercompact cardinals, yet $K(\mathcal{U}_{MAX})$ is not equipped to capture them.

Definition 3.4.8. If $\mathcal{U}$ is a double-index sequence, then $K(\mathcal{U})$, the *core model of* $\mathcal{U}$, is the class of sets constructible from $\mathcal{U}$-mice. More specifically, if M is a $\mathcal{U}$-mouse

then let $A_M \subset \xi_M$ code the Σ_1 theory of M with parameters in $p_M \cup \xi_M$, so that M is constructible from A_M. Then $K(\mathcal{U}) = L[A^{\mathcal{U}}]$, where $A^{\mathcal{U}} = \{(\alpha, \beta, \xi) : \alpha \in A_M$, where M is the βth $\mathcal{U}$-mouse in the $<$-order with $\xi = \xi_M\}$

We now are after a particular sequence $\mathcal{U}_{MAX}$ from which we can build the true core model below $o(\kappa) = \kappa^{++}$. The first property we would like this sequence to have is strongness, i.e. we would like the core model for a sequence $\mathcal{U}$ to actually see that $\mathcal{U}$ is an ultrafilter sequence.

Definition 3.4.9. A double-index sequence $\mathcal{U}$ is *strong* if $\mathcal{U}$ is an ultrafilter sequence in $K(\mathcal{U})$, and $\mathcal{U}$ is *strong above* κ if $\mathcal{U} \restriction \geq \kappa$ is an ultrafilter sequence in $K(\mathcal{U})$.

We now come to the problem of constructing a strong sequence $\mathcal{U}$. We would like to define $\mathcal{U}(\kappa, \beta)$ by recursion on pairs (κ, β), but we need to make sure that when we choose a specific $\mathcal{U}(\kappa, \beta)$, it will still be an ultrafilter in $K(\mathcal{U})$ when all of $\mathcal{U}$ has been defined. The easiest fix to this problem would be to require that $\mathcal{U}(\kappa, \beta)$ be an ultrafilter in V. Unfortunately, this does not suffice. To see why, suppose $0^{\sharp}$ exists, and let U be the derived ultrafilter from an elementary embedding $j : L \to L$. Then U is a measure in $J_{\alpha}[U]$ for some values of α, but is not actually a measure in V, since $0^{\sharp}$ does not imply the existence of measurable cardinals. We would still like to be able to include these partial measures in our study, though, and it is the next definition gives the necessary modification to accomodate for this.

Definition 3.4.10. Let $\mathcal{U}$ be a double-index sequence. Suppose $\kappa \geq \ell(\mathcal{U})$ and U is an ultrafilter on $\mathcal{P}(\kappa) \cap K(\mathcal{U})$. We say U is a $K(\mathcal{U})$-*ultrafilter* if

1. (normality in $K(\mathcal{U})$) if $f \in K(\mathcal{U})$ and $\{\alpha : f(\alpha) < \alpha\} \in U$, then for some $\beta < \kappa$, $\{\alpha : f(\alpha) = \beta\} \in U$.

2. (coherence) $\mathcal{U} = i^U(\mathcal{U}) \restriction \kappa + 1$, and if $f \in K(\mathcal{U})$ and $\{\alpha : f(\alpha) < o^{\mathcal{U}}(\alpha)\} \in U$, then for some $f' \in L[\mathcal{U}], \{\alpha : f(\alpha) = f'(\alpha)\} \in U$.

60

3. (absolute well-foundedness) If $\mathcal{V}$ is any sequence which is strong above κ, and $\mathcal{V} \upharpoonright \kappa + 1 = \mathcal{U}$, then $K(\mathcal{V})^\kappa/U$ is well-founded.

We would like to make a few remarks about Definition 3.4.10. The first note is that the ultrafilter U is on $\mathcal{P}(\kappa) \cap K(\mathcal{U})$, not necessarily all subsets of κ. Hence, the first part of clause (ii) must not be misconstrued: The statement that $i^U(\mathcal{U}) \upharpoonright \kappa + 1 = \mathcal{U}$ means only that $\mathrm{dom}(i^U(\mathcal{U})) = \mathrm{dom}(\mathcal{U})$, and also that for any $(\lambda, \beta) \in \mathrm{dom}(\mathcal{U})$ and for any $x \in \mathcal{P}(\kappa) \cap K(\mathcal{U})$, $x \in \mathcal{U}(\lambda, \beta)$ iff $x \in i^U(\mathcal{U})(\lambda, \beta)$. It is important to note that x comes from $K(\mathcal{U})$, as in general there may be subsets of κ in $\mathrm{Ult}(K(\mathcal{U}), U)$ which are not in $K(\mathcal{U})$, which would mean that $i^U(\mathcal{U})(\kappa, \beta) \supsetneq \mathcal{U}(\kappa, \beta)$. In other words, the ultrapower $K(\mathcal{V})^\kappa/U$ in clause (iii) is to be read literally.

The second remark we would like to make is that the second part of clause (ii) may seem unusual at first. If f and f' are as in clause (ii), then the condition that $[f]_U = [f']_U$ is a stronger coherence condition which ensures that $\mathcal{U} = i^U_{L[\mathcal{U}]}(\mathcal{U}) \upharpoonright \kappa + 1$ as well. No additional functions from $K(\mathcal{U})$ are required to get coherence: it is just a property of the functions constructible from $\mathcal{U}$.

It turns out, however, that the somewhat strange situation outlined immediately after Definition 3.4.10, that $i^U(\mathcal{U})(\kappa, \beta) \supsetneq \mathcal{U}(\kappa, \beta)$, does not happen if $\mathcal{U}$ is strong. Theorem 3.4.11 is what illustrates this, and allows us to build strong sequences. We will not prove it.

Theorem 3.4.11. *1. If $\mathcal{U}$ is strong and U is a $K(\mathcal{U})$-ultrafilter on κ, then for any*

$$F \in {}^\kappa K(\mathcal{U}) \cap K(\mathcal{U}), \ \{\eta < \kappa : F(\xi) \in U\} \in K(\mathcal{U}).$$

2. If $\mathcal{U}$ is a sequence such that each $\mathcal{U}(\kappa, \beta)$ is a $K(\mathcal{U} \upharpoonright (\kappa, \beta))$-ultrafilter, then $\mathcal{U}$ is strong.

Remark 3.4.12. The conclusion of the first clause of Theorem 3.4.11 was first used by Kunen in [7] and, confusingly, is what is often referred to as a "$K(\mathcal{U})$-ultrafilter" in the literature.

61

We have finished our in-depth discussion of the first property, i.e. strongness, that we wish for this particular sequence $\mathcal{U}_{MAX}$ to have. The second property we would like it to have is maximality. Intuitively, a maximal sequence is one which contains measures on all measurable cardinals in V.

Definition 3.4.13. If κ is an ordinal, then $\mathcal{U}$ is said to be *maximal at* κ if there is no strong sequence $\mathcal{U}'$ with $\mathcal{U}' \restriction \kappa = \mathcal{U} \restriction \kappa$ and $o^{\mathcal{U}'}(\kappa) > o^{\mathcal{U}}(\kappa)$. A sequence $\mathcal{U}$ is *maximal* if it is maximal at every ordinal κ, i.e. for no κ is there a $K(\mathcal{U} \restriction \kappa + 1)$-ultrafilter.

Notice that if $\mathcal{U}'$ witnesses that $\mathcal{U}$ is not maximal then $\mathcal{U}' \restriction (\kappa, o^{\mathcal{U}}(\kappa)) = \mathcal{U} \restriction (\kappa+1)$ and so $U = \mathcal{U}'(\kappa, o^{\mathcal{U}}(\kappa))$ is a $K(\mathcal{U} \restriction (\kappa + 1))$-ultrafilter on κ.

At long last, we now have the requisite information to define the sequence $\mathcal{U}_{MAX}$.

Theorem 3.4.14. *There is a maximal sequence $\mathcal{U}_{MAX}$ which is strong and unique.*

Proof. We define the sequence $\mathcal{U}_{MAX}$ by recursion on pairs (κ, β): If $\mathcal{U}_{MAX} \restriction (\kappa, \beta)$ is defined, then set $\mathcal{U}_{MAX}(\kappa, \beta)$ equal to any $K(\mathcal{U}_{MAX} \restriction (\kappa, \beta))$-ultrafilter, if any exists, and otherwise set $o^{\mathcal{U}}(\kappa) = \beta$. That $\mathcal{U}_{MAX}$ is maximal, strong, and unique is omitted. $\square$

Finally, the model $K(\mathcal{U}_{MAX})$ is the canonical model below $o(\kappa) = \kappa^{++}$:

Definition 3.4.15. The *core model below* $o(\kappa) = \kappa^{++}$ is the model $K(\mathcal{U}_{MAX})$.

3.5   A STROLL THROUGH K

As promised, we will now pause and give a few examples to illustrate these ideas. Intuitively, for each possibility of the large cardinal structure in V below $o(\kappa) = \kappa^{++}$, there will be different mice, a different $\mathcal{U}_{MAX}$ which contains all of the measures on measurable cardinals in V, and a different $K(\mathcal{U}_{MAX})$. We will illustrate what the picture looks like as the large cardinal hierarchy is climbed.

Example 3.5.1. Suppose $V = L$. Then the only mice we can build are the simplest ones, obtained by taking $\xi = \alpha$, giving us the 0-mice which are just the levels of the J-hiearchy. The sequence $\mathcal{U}_{MAX}$ is empty since L contains no measurable cardinals, and $K(\mathcal{U}_{MAX}) = L$. That the empty sequence is strong, maximal, and unique are all easy to check.

Example 3.5.2. Suppose $V = L[0^\sharp]$. Then we may build nonconstructible mice as follows. Since $0^\sharp$ exists, there is a nontrivial elementary embedding $j : L \to L$ with critical point κ. Let U be the derived ultrafilter from j, i.e.

$$U = \{X \subseteq \kappa : X \in L \text{ and } \kappa \in j(X)\}.$$

First, we claim that we may satisfy Definition 3.4.4 with $p = 0, \xi = \kappa, \mathcal{U} = 0, \mathcal{V}$ equal to the sequence with a single entry $U = \mathcal{V}(\kappa, 0)$, and $\alpha = (\kappa^+)^L$. Clause 1 and 2 are clearly satisfied. By Lemma 1.5.4, this α suffices to satisfy the part of condition 3 requires U be an ultrafilter in $J_\alpha[\mathcal{U}]$. To see that $J_{(\kappa^+)^L}[U]$ satisfies the iterability condition of clause 3, we refer the reader to [13]. Hence, $N = J_{(\kappa^+)^L}[U]$ is a 0-mouse.

This is not the smallest possible 0-mouse, though. We claim that we may choose $\xi = 0$ instead of $\xi = \kappa$. If we choose this, then notice that we cannot let $\mathcal{V} = U$: by condition 3, we must choose an α large enough so that U is an ultrafilter sequence in $J_\alpha[U]$, and for no such α do we have condition 1 satisfied, since $J_\alpha[U] \neq \mathcal{H}^1(\emptyset)$ as computed in $J_\alpha[U]$. We must take the transitive collapse of the hull instead.

Motivated by this observation, let α be the ordinal so that if

$$\pi : M \cong H \prec_1 J_{(\kappa^+)^L}[U]$$

is the collapse of the Σ_1-hull of the empty set as computed in $J_{(\kappa^+)^L}[U]$, then $M = J_\alpha[\pi^{-1}[U]]$. Then M is a 0-mouse with this ordinal α and $\mathcal{V} = \pi^{-1}[U]$. This is the smallest nonconstructible mouse, and is often called $\mathcal{M}_0^\sharp$ in the literature instead of M. By Lemma 3.4.7, N is actually an iterated ultrapower of M, and we also have that

the only mice in V are 0-mice, which are the levels of J as in the previous example, M, and iterates of M.

One may be led to believe since $\mathcal{U}_{MAX}$ should contain all measures below $o(\kappa) = \kappa^{++}$, that $\pi^{-1}[U]$ should be in $\mathcal{U}_{MAX}$. This is not true, though, since of course $L[0^\sharp]$ does not see that $\pi^{-1}[U]$ is an $L[0^\sharp]$-ultrafilter (See Definition 3.4.10). Thus, since V does not contain any ultrafilter sequences, $U_{MAX} = \emptyset$. That $\emptyset$ is strong and maximal is trivial, and by Definition 3.4.15, $K(\emptyset)$ is the class of sets constructible from 0-mice, which is $L[0^\sharp]$, as we would expect from our intuitive idea of what the core model should be.

Example 3.5.3. Suppose $V = L[(0^\sharp)^\sharp]$. Then the way we build an additional mouse works similarly as in the previous example. By Lemma 1.5.5, there exists a nontrivial elementary embedding $j : L[0^\sharp] \to L[0^\sharp]$. Let $\kappa = \operatorname{crit}(j)$ and let U be the derived ultrafilter from j, i.e.

$$U = \{X \subseteq \kappa : X \in L[0^\sharp] \text{ and } \kappa \in j(X)\}.$$

Then there are multiple ξ's which will satisfy the conditions of Definition 3.4.4, just as in the previous example, but we will focus on the minimal ξ. We may let $p = 0$, $\xi = 0, \mathcal{U} = 0, \mathcal{V} = U$, and let α be the ordinal so that if

$$\pi : M \cong H \prec_1 J_{(\kappa^+)^{L[0^\sharp]}}[U]$$

is the collapse of the Σ_1-hull of the empty set as computed in $J_{(\kappa^+)^L}[U]$, then $M = J_\alpha[\pi^{-1}[U]]$. Then, exactly as in the previous example, M is a 0-mouse, which is often referred to as $\mathcal{M}_0^{\sharp\sharp}$ in the literature. We still refer the reader to [13] for the question of iterability.

The reader should note that we may not let the filter from the previous example (derived from an embedding $j : L \to L$) be part of the sequence $\mathcal{U}$, since $L[(0^\sharp)^\sharp]$ does not see this filter as an $L[(0^\sharp)^\sharp]$-ultrafilter. It is also important to note that, before we

take a transitive collapse or Skolem hull, we index $L[U]$ up to $(\kappa^+)^{L[0^\sharp]}$ rather than $(\kappa^+)^L$. This pattern will persist all the way up through $o(\kappa) = \kappa^{++}$, where we index a mouse M with measurable cardinal κ at $J_\alpha[U]$, where α is κ^+ as computed in the 'previous' model.

Example 3.5.4. Define $(0^\sharp)^\alpha$ inductively on α by setting $(0^\sharp)^0 = \emptyset$, $(0^\sharp)^{\alpha+1} = ((0^\sharp)^\alpha))^\sharp$, and for limit ordinals γ let $(0^\sharp)^\gamma$ be a canonical set of ordinals encoding $\langle (0^\sharp)^\alpha : \alpha < \gamma \rangle$. If α is a successor ordinal, one may construct the 0-mouse $\mathcal{M}_0^{\sharp\alpha}$ just as in the previous two examples, except we take the transitive collapse of the Skolem hull of the model $J_{(\kappa^+)^{L[(0^\sharp)^{\alpha-1}]}}[U]$, κ is the critical point of an embedding $j : L[(0^\sharp)^{\alpha-1}] \to L[(0^\sharp)^{\alpha-1}]$, and U is the ultrafilter derived from j.

If α is a limit ordinal, notice that there is no way to build a 0-mouse in this way, since assuming $(0^\sharp)^\alpha$ exists does not imply the existence of any elementary embeddings of the form $j : L[X] \to L[X]$ which has not already come from a previous $(0^\sharp)^\beta$. There is no choice of the sequence $\mathcal{V}$ which would give any new mice. Hence, we only have 0-mice corresponding to $(0^\sharp)^\alpha$ for successor ordinals α.

One may build all possible $(0^\sharp)^\alpha$ in this way until we have built the model $M = L[\langle (0^\sharp)^\alpha : \alpha \in \mathrm{Ord} \rangle]$, which is intuitively some 'minimal' model for the property "$X^\sharp$ exists for every set X," and once it may seem like we have exhausted all the sharps, we may construct the 'sharp' for this large cardinal property, i.e. we may assume there is an elementary embedding

$$j : L[\langle (0^\sharp)^\alpha : \alpha \in \mathrm{Ord} \rangle] \to L[\langle (0^\sharp)^\alpha : \alpha \in \mathrm{Ord} \rangle],$$

letting $\kappa = \mathrm{crit}(j)$, letting U be the ultrafilter derived from j, and building yet another 0-mouse above each $\mathcal{M}_0^{\sharp\alpha}$ by taking the transitive collapse of the appropriate Skolem hull of $J_{(\kappa^+)^{L[\langle (0^\sharp)^\alpha : \alpha \in \mathrm{Ord} \rangle]}}[U]$.

One can predict that in all of these situations, $\mathcal{U}_{\mathrm{MAX}}$ is still empty and that $K(U_{\mathrm{MAX}}) = \bigcup \{x : x \text{ is a 0-mouse}\}$, however many of those exist in V.

We now begin to describe mice further up the large cardinal hierarchy. At this point, we will stop beginning each example with a "$V = M$"-type assertion, since the way the mice are constructed does not depend on V— only the sequence $\mathcal{U}_{\mathrm{MAX}}$ and the model $K(\mathcal{U}_{\mathrm{MAX}})$ do. The core model becomes wildly complicated once actual measurable cardinals, particularly of higher Mitchell order, are involved, so we will describe the process by which mice are constructed, rather than specifically listing every single mouse and the $p, \xi, \mathcal{U}, \mathcal{V}$, and α values associated with each one.

Example 3.5.5. Suppose κ is a measurable cardinal in V and U is a measure on κ. Then, just as in Example 3.5.1, the models $J_\alpha[U]$ for $\alpha \geq \kappa^+$ are U-mice. The iterability condition is satisfied by Lemma 3.1.6.

This example provides a case where the least possible value of ξ is not necessarily the Σ_1-projectum: Let $M = L_{(\kappa^+)^L}[U]$. Then $\xi_M = \kappa$ since $\mathcal{H}^1(\kappa)$ as computed in M is just M, yet $\varrho_1^M = \omega$ since the Σ_1-theory of M with no parameters is a new real.

Now we give our first examples of mice above $L[U]$. The most common method of constructing mice is a generalization of what we have done in the previous examples. It is easy to see for sufficiently large α that $J_\alpha[\mathcal{U}]$ is a $\mathcal{U}$-mouse for any coherent sequence of measures $\mathcal{U}$. Using this observation, suppose $J_\alpha[\mathcal{U}]$ is a $\mathcal{U}$-mouse, $x \in J_\alpha[\mathcal{U}]$, and $x \subseteq \xi$ where, above ξ, $\mathcal{U}$ is an ultrafilter sequence in $J_\alpha[\mathcal{U}]$ all of whose iterated ultrapowers are well-founded and $o^{\mathcal{U}}(\xi) = 0$. Let M be the transitive collapse of the Σ_1 Skolem hull of $\xi \cup p$ as computed in $J_\alpha[\mathcal{U}]$. Then M is a $\mathcal{U} \upharpoonright \xi$-mouse with $x \in M$. The key takeaway, informally, is that if we already have a $\mathcal{U}$-mouse and $\mathcal{U}'$ is an initial segment of $\mathcal{U}$, then it should not be too hard to build a $\mathcal{U}'$-mouse. We illustrate this in the next example. Additionally, we leave open the question of iterability (and exactly what iterated ultrapowers of more complicated mice precisely are) to Mitchell's original papers [9], [10], and [11].

Example 3.5.6. Suppose there are two measures U and W on measurable cardinals $\kappa < \lambda$, and let $N = J_{(\lambda^+)^{L[U,W]}}[U, W]$. It is easy to see that N is a $\langle U, W \rangle$-mouse

with $\mathcal{U} = \mathcal{V} = \langle U, W \rangle$ so that $\ell(\mathcal{U}) = \lambda^+$, since no transitive collapse or Skolem hull are necessary. This example also shows that we may have $\mathcal{U} = \mathcal{V}$, even though Definition 3.4.4 makes it seem like $\mathcal{U}$ must always be an initial segment of $\mathcal{V}$. In this case, though, $\xi = (\lambda^+)^{L[U,W]}$ and so actually $\mathcal{U} = \mathcal{V} \upharpoonright (\lambda^+)^{L[U,W]}$. This is why $\xi_N = (\lambda^+)^{L[U,W]}$.

Now that we have a $\langle U, W \rangle$-mouse, we may build $\mathcal{U}'$-mice where $\mathcal{U}'$ is a proper initial segment of $\langle U, W \rangle$. In this paragraph we build a 0-mouse this way, and in the next paragraph we build a U-mouse this way. First, we claim that we may construct the minimal mouse above $L[U]$, $\mathcal{M}_0^\dagger$, in the same way we constructed $\mathcal{M}_0^\sharp$ as the minimal mouse above L. Furthermore, this mouse is actually also a 0-mouse. First note that we would need $\xi = \omega$ since $0^\dagger$ is a subset of ω. By further letting $p = 0$, set M equal to the transitive collapse of the Σ_1-hull of ω as computed in N, and the structure M is a $\langle U, V \rangle \upharpoonright \omega$-mouse (i.e. just a 0-mouse) which corresponds to $0^\dagger$ in the same way the M from Example 3.5.2 corresponded to $0^\sharp$.

We now restrict the sequence $\mathcal{U} = \langle U, W \rangle$ in a different way. Take some ξ strictly between κ^+ and λ, and let p be a sequence of length 1 which is any ordinal between κ^+ and λ. Let M be the transitive collapse of the Skolem hull of $\xi \cup p$ as computed in N. Then M is a U-mouse with $\mathcal{V} = \langle U, \pi^{-1}(W) \rangle$. Since the critical point is at least ξ, $M \vDash$ "There are two measurable cardinals," but M is only a U-mouse now, as ξ is below where the second measure is indexed. This example is illustrated in Figure 3.1.
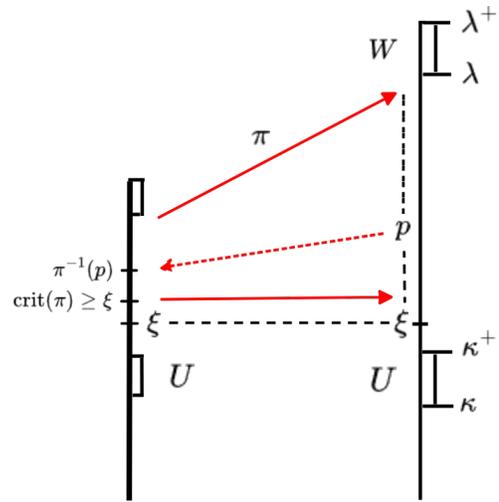
Figure 3.1: On the right, the trivial $\langle U, W \rangle$-mouse $N = L_{\lambda^+}[U, W]$, its corresponding ξ and p-values, and the measures U and W. On the left, the U-mouse $M = \pi^{-1}(\mathcal{H}^1_N(\xi \cup p))$, along with the measure U and the 'fake' measure $\pi^{-1}[W]$. The map $\pi^{-1} : N \to M$ fixes ξ and U, but moves p and W.

3.6 THE LOWER BOUND

Now that we have defined the core model $K(\mathcal{U}_{MAX})$, we may prove some of its basic properties under the assumption that there is no model of $\exists\kappa\ o(\kappa) = \kappa^{++}$ in preparation for the lower bound that we have been after. From this point forward, we write "K" for the model $K(\mathcal{U}_{MAX})$, and we will write $K(\mathcal{U})$ when referring to the core model for a different sequence $\mathcal{U}$, as in Definition 3.4.8.

The following lemma is the main lemma used in the proof of the lower bound, and is one of the most important and widely used properties of K, namely that every elementary embedding from K is actually an iterated ultrapower. The details of the proof would require more sophisticated fine structural analysis than we assume knowledge of in this thesis, and so we sketch the proof and refer the reader to [11] for the details.

Lemma 3.6.1. If there is no model of $\exists\kappa\ o(\kappa) = \kappa^{++}$, then every embedding $i : K \to N$ into a well-founded model N is an iterated ultrapower by measures in $\mathcal{U}_{MAX}$.

Proof sketch: We first prove a weaker claim.

Claim 3.6.1. If $\mathcal{U}$ is any strong full[4] sequence, then there is an iterated ultrapower $i : K \to K(\mathcal{U})$.

---

[4] *Fullness* is a technical condition imposed on the sequence $\mathcal{U}$ which is only used here. If τ is a regular cardinal, then $\mathcal{U}$ is τ-full if there is a τ-closed, unbounded class C of ordinals such that if $\nu \in C$ then $(\nu^+)^{K(\mathcal{U})} = \nu^+$. We define it here to reduce clutter in the main text, since this condition is only used in this claim. All sequences in which this claim, or Lemma 3.6.2, are applied will be full.

Proof sketch: The iterated ultrapower i is defined recursively. Suppose $i_\nu : K \rightarrow K(\mathcal{U}_\nu)$ has been defined. If $\mathcal{U}_\nu = \mathcal{U}$ then set $i = i_\nu$. Otherwise, find some a_ν such that $o^{\mathcal{U}_\nu}(a_\nu) \neq o^{\mathcal{U}}(a_\nu)$, and if $b_\nu = \inf(o^{\mathcal{U}_\nu}(a_\nu), o^{\mathcal{U}}(a_\nu))$ then $\mathcal{U}_\nu \upharpoonright (a_\nu, b_\nu) = \mathcal{U} \upharpoonright (a_\nu, b_\nu)$. It can be shown that in this case, we have that $b_\nu = o^{\mathcal{U}_\nu}(a_\nu) < o^{\mathcal{U}}(a_\nu)$, and so $i_{\nu,\nu+1}$ is defined to be the ultrapower by the measure $\mathcal{U}_\nu(a_\nu, b_\nu)$. The reader is referred to [11] for the details of the proof of the claim. $\square$

Notice that it suffices to prove the lemma under the assumption that i is set based, i.e. that there is an α such that $N = \{i(f)(\beta) : f \in K \text{ and } \beta < \alpha\}$. The complete lemma follows from this, for if i is arbitrary then the maps i_α,

$$i_\alpha : K \rightarrow N_\alpha \cong \{i(f)(\beta) : f \in K \text{ and } \beta < \alpha\} \prec N$$

are all set based and hence iterated ultrapowers. The initial segments of the iterations i_α will fit together to yield i.

By elementarity we see that $N = K(i(\mathcal{U}_{\mathrm{MAX}}))$. It must be verified that $i(\mathcal{U}_{\mathrm{MAX}})$ satisfies the hypotheses of the claim, hence there is an iterated ultrapower $j : K(\mathcal{U}_{\mathrm{MAX}}) = K(i(\mathcal{U}_{\mathrm{MAX}}))$, and all that remains to show need to show that $i = j$, which we refer the reader to [11]. $\square$

Lemma 3.6.2. If U is a normal K-ultrafilter on κ and $\mathrm{Ult}(K, U)$ is well-founded then $U \in K$. If $\mathrm{crit}(U) > \omega_2$ then the hypothesis that $\mathrm{Ult}(K, U)$ is well-founded is unnecessary.

Proof. See [12]. $\square$

The following theorem is the lower bound of the equiconsistency result we are after.

Theorem 3.6.3. *If there is a measurable cardinal κ with $2^\kappa > \kappa^+$, then there exists an inner model with a measurable cardinal λ such that $o(\lambda) = \lambda^{++}$.*

70

Proof. Towards a contradiction, suppose κ is measurable and $2^\kappa > \kappa^+$, but there is no inner model with a measurable cardinal λ such that $o(\lambda) = \lambda^{++}$.

Let U be any measure on κ and let $i_U : V \to M = \mathrm{Ult}(V, U)$ be the usual ultrapower embedding. First, we claim that $|i_U(\kappa)| = 2^\kappa$. We have that $|i_U(\kappa)| \leq 2^\kappa$ since any ordinal $\alpha < i_U(\kappa)$ is represented by a function $f : \kappa \to \kappa$, of which there are 2^κ. To see that $i_U(\kappa) \geq 2^\kappa$, notice that κ is measurable in V, so $i_U(\kappa)$ is measurable in M. In particular, $i_U(\kappa)$ is a strong limit cardinal in M. Let μ be any cardinal less than $(2^\kappa)^V$. Then μ is also less than $(2^\kappa)^M$. Hence any V-cardinal less than 2^κ is not strong limit in M, so $i_U(\kappa) \geq 2^\kappa$.

Now, let $i = i_U \restriction K : K \to K^M$ be the restriction of the embedding i_U to K. By Lemma 3.6.1 is an iterated ultrapower of K, so let $\langle N_\nu : \nu \leq \theta \rangle$ be the iterates, so that $N_0 = K$ and $N_\theta = K^M$. If $\nu < \theta$ is a limit ordinal then there are $\xi_\nu < \nu$ and $U_\nu \in N_{\xi_\nu}$ such that $N_{\nu+1} = \mathrm{Ult}(N_\nu, i_{\xi_\nu, \nu}(U_\nu))$. (The reader should be careful and notice that $U_\nu \notin N_\nu$, but $i_{\xi_\nu, \nu}(U_\nu) \in N_\nu$, so that $N_{\nu+1}$ is actually the ultrapower of N_ν by a measure in N_ν.) In addition to ξ_ν, we write κ_ν for the ordinal $i_{0,\nu}(\kappa)$. We first claim that this iteration is of length at least κ^{++}.

Claim 3.6.2. $\theta \geq \kappa^{++}$.

Proof. We prove the claim by bounding the size of $i_{0,\nu}(\kappa)$ for $\nu < \kappa^{++}$ by considering how N_ν is represented in terms of the extender of the iteration at that point. In particular,

$$i_{0,\nu}(\kappa) = \{[a, f] : a \in \{\kappa_\eta : \eta < \nu\}^{<\omega} \text{ and } f : [\kappa]^{<\omega} \to \kappa, \ f \in K\}.$$

By simple cardinal arithmetic, $|\{\kappa_\eta : \eta < \nu\}^{<\omega}| = |\nu| < \kappa^{++}$ and $\kappa^{(\kappa^{<\omega})} = \kappa^+$, so $|i_{0,\nu}(\kappa)| \leq |\nu| \cdot \kappa^+ < \kappa^{++}$. But we showed previously that $|i(\kappa)| = |i_{0,\theta}(\kappa)| = 2^\kappa$, but since the GCH fails at κ, $|i(\kappa)| = |i_{0,\theta}(\kappa)| > \kappa^+$, a contradiction. $\square$

Claim 3.6.3. There is a stationary class $\Gamma \subseteq \kappa^{++}$ of ordinals of cofinality ω such

that $\xi_\nu = \bar\xi$ and $U_\nu = \bar U$ are constant for $\nu \in \Gamma$.

Proof. The function $\nu \mapsto \xi_\nu$ is regressive on limit points of κ^{++} of cofinality ω, so by Fodor's Lemma there is some stationary $\Gamma' \subset \mathrm{Lim}(\kappa^{++})$ of points of cofinality ω on which $\nu \mapsto \xi_\nu$ is constant. Denote this constant value $\bar\xi$.

Let

$$\Phi : \Gamma' \to N_{\bar\xi}$$

be the function which sends an ordinal ν to the measure U_ν. We know that every measure in the iteration has critical point at most $i(\kappa)$, so $\mathrm{crit}(\Phi(\nu)) \leq \kappa_{\bar\xi}$ for every $\nu \in \Gamma'$. We now wish to find a bound for the size of $\mathrm{ran}(\Phi)$, and we do this by counting the number of measures on cardinals in $N_{\bar\xi}$ less than or equal to $\kappa_{\bar\xi}$. Since $N_{\bar\xi} \models |o(\kappa_{\bar\xi})| \leq \kappa_{\bar\xi}^+$ by our contradiction assumption, and $(\kappa_{\bar\xi}^+)^{N_{\bar\xi}} < \kappa^{++}$, in V we can see that $|\mathrm{ran}(\Phi)| \leq \kappa^+$. Let $\langle a_\beta : \beta < \alpha \rangle$ where $\alpha \leq \kappa^+$ be a well-ordering of $\mathrm{ran}(\Phi)$. Now let

$$\Psi : \Gamma' \to \alpha$$

be the function so that $\Phi(\nu) = a_{\Psi(\nu)}$. Once again, we may use Fodor's lemma to find a stationary $\Gamma \subseteq \Gamma'$ on which Ψ is constant. Denote this constant value $\bar a$, and let η be the ordinal so that $\bar a = a_{\Psi(\eta)}$. Then the measure $\bar U = \Phi(\eta)$, the stationary class Γ, and the ordinal $\bar\xi$ are as desired. This completes the proof of the Claim. $\square$

Now fix a particular $\nu \in \Gamma \cap \mathrm{Lim}(\Gamma)$. Let $\vec\kappa = \langle \nu_n : n < \omega \rangle$ be a cofinal sequence in $\Gamma \cap \nu$, and for each $n < \omega$ let $\kappa^{(n)} = \mathrm{crit}(i_{\nu_n, \nu})$.

Claim 3.6.4. $\vec\kappa$ generates the measure $i_{\bar\xi, \nu}(\bar U)$.

Proof. Work in N_ν. Then we have $A \in i_{\bar\xi, \nu}(\bar U)$ if and only if there is some $n < \omega$ and $\bar A \in N_{\nu_n}$ such that $i_{\nu_n, \nu}(\bar A) = A$ and $\bar A \in i_{\bar\xi, \nu_n}(\bar U)$. Equivalently, $\kappa^{(n)} \in i_{\nu_n, \nu_{n+1}}(\bar A)$ for some $n < \omega$, i.e. $\{\kappa_n, \kappa_{n+1}, \kappa_{n+2}, ...\} \in A$ for some $n < \omega$.

$\square$

Note that $K^M|(\kappa_\nu^+)^{K^M} = N_\nu|(\kappa_\nu^+)^{N_\nu}$, so $i_{\bar\xi,\nu}(\bar U)$ is still a K^M-ultrafilter.

So, since $^\omega M \subseteq M$, we have that $i_{\bar\xi,\nu}(\bar U) \in M$. Since $\text{crit}(i_{\bar\xi,\nu}) > \omega_2$, Lemma 3.6.2 implies that $i_{\bar\xi,\nu}(\bar U) \in K^M = N_\theta$. But then $i_{\bar\xi,\nu}(\bar U) \in N_{\nu+1}$, which is a contradiction. Hence, there is an inner model with a measurable cardinal λ such that $o(\lambda) = \lambda^{++}$, which completes the proof of Theorem 3.6.3. $\square$

Remark 3.6.4. The reader should note that crucial usage of the hypothesis that the GCH fails at κ occurrs in Claim 3.6.2. Since $2^\kappa > \kappa^+$ in V but the GCH holds in K, the measure U must be measuring subsets of κ that are not in K. If the GCH holds at κ, then the iteration $i : K \to K^M$ may be only one step, say, if $V = L[U]$.

To go even further, note that the iteration being of length at least κ^{++} specifically is crucial: suppose $\theta = \kappa^+$. Then the proof of Claim 3.6.3 does not go through, since while we can still take a stationary $\Gamma \subseteq \kappa^+$, we are doomed when we must define the map Ψ, since Γ' would have size κ^+ and $\alpha \leq \kappa^+$, so we have no way of knowing that Ψ is regressive.

BIBLIOGRAPHY

[1] James E. Baumgartner. Iterated forcing. Surveys in Set Theory, volume 87 of London Mathematical Society Lecture Note Series, Adrian R.D. Mathias, editor, 1–59. Cambridge University Press, Cambridge, 1983.

[2] James W. Cummings, *Iterated Forcing and Elementary Embeddings* Handbook of Set Theory, M. Foreman and A. Kanamori, editors, Springer, Dordrecht, 2010

[3] Sy-David Friedman and Katherine Thompson, Perfect trees and elementary embeddings, The Journal of Symbolic Logic 73(3):906–918, 2008.

[4] Moti Gitik. The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. Annals of Pure and Applied Logic, 43(3):209–234, 1989.

[5] Thomas J. Jech. Set Theory. The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer, Berlin, 2002.

[6] Akihiro Kanamori. The Higher Infinite. Springer Monographs in Mathematics. Springer, Berlin, 2003. Second edition.

[7] Kenneth Kunen. Some applications of iterated ultrapowers in set theory. Annals of Mathematical Logic, 1(2):179-227, 1970.

[8] Kenneth Kunen. Set Theory. An Introduction to Independence Proofs. North-Holland, Amsterdam, 1983. Reprint of the 1980 original.

[9] William J. Mitchell. The core model for sequences of measures. I. Mathematical Proceedings of the Cambridge Philosophical Society, 95(2):229-260, 1984.

[10] William J. Mitchell. The core model for sequences of measures. II. Unpublished, retrieved from https://people.clas.ufl.edu/wjm/publications/

[11] William J. Mitchell. The core model for sequences of measures. III. Unpublished, retrieved from https://people.clas.ufl.edu/wjm/publications/

[12] William J. Mitchell, *The Covering Lemma*, Handbook of Set Theory, M. Foreman and A. Kanamori, editors, Springer, Dordrecht, 2010

[13] Ernest Schimmerling. The ABC's of mice. Bull. Symbolic Logic, 7(4):485–503, 2001.

[14] John R. Steel, *An Outline of Inner Model Theory* Handbook of Set Theory, M. Foreman and A. Kanamori, editors, Springer, Dordrecht, 2010

[15] John R. Steel. A Comparison Process for Mouse Pairs. Cambridge University Press, 2022.

[16] Martin Zeman, Inner Models and Large Cardinals, De Gruyter Series in Logic and its Applications, vol. 5, Walter De Gruyter, Berlin, 2001